AQA Board. **GCE Maths (Specification A).**

Model Answers

June 2003

Note comment on paper about stating final answers to 3 significant figures, in general

Pure 1 - MAP1, 3 June 2003

Question 1

(a) (i)

$$\int x^{\frac{3}{2}}\ dx = \frac{2}{5}.x^{\frac{5}{2}} + C$$

(ii)

$$\int_0^4 x^{\frac{3}{2}} = \frac{2}{5}\left[x^{\frac{5}{2}}\right]_0^4$$

$$= \frac{2}{5}(32 - 0) = 12.8$$

(b) Area under OP between 0 and 4 = 16.

So the area of the shaded region

$$= 16 - 12.8 = 3.2$$

Question 2

$$y = x + 4x^{-2}$$

(a)

$$\frac{dy}{dx} = 1 - 8x^{-3}$$

(b) At stationary point

$$1 - 8x^{-3} = 0$$

$$x^3 = 8$$

$$x = 2$$

so required coordinates are

$$(2, 3)$$

(c)

$$\frac{d^2y}{dx^2} = 24x^{-4}$$

At the stationary point (2,3)

$$\frac{d^2y}{dx^2} = +1.5$$

so this stationary point is a minimum

Question 3

(a) (i)

$$\text{Area of sector } = \frac{1}{2}r^2\theta$$

so

$$\frac{1}{2}.8^2\theta = 32\ \theta\ cm^2$$

(ii)

$$Area = \frac{1}{2}(\text{base} \times \text{height})$$

$$= \frac{1}{2} \times 8 \times 8\sin\theta$$

$$= 32\sin\theta\ cm^2$$

(iii) Area of segment

$$= 32\theta - 32\sin\theta$$

(b)

$$8\theta - 8\sin\theta - \pi = 0$$

(i) When $\theta = 1.3$, expression becomes

$$8 \times 1.3 - 8\sin 1.3 - \pi = -0.450$$

When $\theta = 1.4$, expression becomes

$$8 \times 1.4 - 8\sin 1.4 - \pi = 0.175$$

Since the expression changes sign within these values of x, then there must be a root between them.

(ii)

$$f(1.4) - f(1.3) \approx 0.625$$

so increment to 1.3 would be

$$\approx \frac{0.450}{0.625} \approx 0.720$$

giving an estimate for the root of

$$1.37$$

Question 4

(a)

$$2\cos^2 x = 2 + \sin x$$

$$2(1 - \sin^2 x) = 2 + \sin x$$

$$2\sin^2 x + \sin x = 0$$

(b) above eqn. becomes

$$\sin x(2\sin x+1)=0$$

therefore

$$\sin x=0\Rightarrow x=0,\pi$$

or

$$2\sin x=-1\Rightarrow \sin x=-\frac{1}{2}\Rightarrow x=\frac{7\pi}{6},\frac{11\pi}{6}$$

Question 5

$$f(x)=2+\ln x$$

(a) (i)

$$f'(x)=\frac{1}{x}$$

(ii) When x=e

$$f'(x)=\frac{1}{e}$$

(b) translation by 2 units in positive y direction

(c) (i) Range is given by

$$-\infty\le y\le\infty$$

(ii) Domain is given by

$$-\infty\le x\le\infty$$

Range given by

$$y>0$$

(iii)

$$y=2+\ln x$$

$$\ln x=y-2$$

$$x=e^{y-2}$$

stated otherwise

$$f^{-1}(y)=e^{y-2}$$

altering variables gives

$$f^{-1}(x)=e^{x-2}$$

(d) (i)

$$fg(x)=f(ex^3)=2+\ln(ex^3)$$

$$=2+\ln e+\ln x^3=2+1+3\ln x$$

$$=3+3\ln x=3(1+\ln x)$$

(ii)

$$fg(x)=9$$

$$3(1+\ln x)=9$$

$$\ln x=2$$

$$x=e^2$$

Question 6

(a) (i) First term is

$$a$$

Second term is

$$a+a\left(\frac{p}{100}\right)$$

$$=a\left(1+\frac{p}{100}\right)$$

and General Term is

$$x+x\left(\frac{p}{100}\right)$$

$$x\left(1+\frac{p}{100}\right)$$

showing the common ratio

(ii) If a = 2000

$$b=2000\left(1+\frac{p}{100}\right)$$

$$c=2000\left(1+\frac{p}{100}\right)^2$$

(b) (i)

$$2000\left(1+\frac{p}{100}\right)^2=2332.80$$

$$p=100\left(\sqrt{\frac{2332.8}{2000}}-1\right)$$

$$p=8$$

(ii) Formula for general term is

$$u_n=ar^{n-1}$$

when the first term is u_1. But here we designate the first term to be u_0 so general term becomes

$$u_n=ar^n$$

so

$$u_n=2000\left(1+\frac{8}{100}\right)^n$$

$$=2000\,(1.08)^n$$

(iii) After 10 years, balance

$$=2000\times 1.08^{10}=\pounds 4317.85$$

Pure 1 - MAP2, 13 June 2003

Question 1

$$\int_0^{\frac{1}{2}} xe^{2x}\,dx$$

$$= \left[x.\frac{e^{2x}}{2}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{e^{2x}}{2}\,dx$$

$$= \left(\frac{1}{4}e\right) - (0) - \left[\frac{e^{2x}}{4}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{4}e - \left\{\frac{e}{4} - \frac{1}{4}\right\}$$

$$= \frac{1}{4}$$

Question 2

(a) 2

(b) (i) -p

(ii)

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

$$= (-p)^2 - 2(2)$$

$$= p^2 - 4$$

(c) From above

$$p^2 - 4 = 5$$

$$p^2 = 9$$

$$p = \pm 3$$

Question 3

(a)

$$\tan(45^\circ + \theta) = \frac{\tan 45 + \tan\theta}{1 - \tan 45 \tan\theta}$$

$$= \frac{1+\tan\theta}{1-\tan\theta}$$

(b)

$$\tan 105^\circ = \tan(45^\circ + 60^\circ)$$

from above

$$= \frac{1+\tan 60}{1-\tan 60}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

multiply top and bottom by $1+\sqrt{3}$

$$= \frac{1+3+2\sqrt{3}}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= -2-\sqrt{3}$$

Question 4

$$2\cos x - \frac{1}{x} = 0$$

(a) When x = 0.6, the LHS of above eqn.

$$= 2\cos 0.6 - \frac{1}{0.6}$$

$$= -0.016$$

When x=0.7, the LHS of above eqn.

$$= 2\cos 0.7 - \frac{1}{0.7}$$

$$= 0.101$$

Since it changes sign in the interval, there is a root in this interval

(b) Newton-Raphson method is given by

$$x_{n+1} = x_n - \frac{f_n}{f'_n}$$

derivative of function is

$$-2\sin x + \frac{1}{x^2}$$

If $x_1 = 0.6$

$$x_2 = 0.6 - \frac{-0.016}{-2\sin 0.6 + \frac{1}{0.6^2}}$$

$$x_2 = 0.6 - \frac{-0.016}{1.648}$$

$$x_2 = 0.610$$

Question 5

$$y = \frac{2x}{\sin x}$$

(a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x(2) - 2x\cos x}{\sin^2 x} \\ &= \frac{2\sin x - 2x\cos x}{\sin^2 x}\end{aligned}$$

(b) (i) The gradient at $\left(\frac{\pi}{2}, \pi\right)$

$$\begin{aligned}&= \frac{2\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}} \\ &= \frac{2}{1} = 2\end{aligned}$$

The eqn of the tangent is

$$y - y_0 = 2(x - x_0)$$

at $\left(\frac{\pi}{2}, \pi\right)$

$$\begin{aligned}y - \pi &= 2(x - \frac{\pi}{2}) \\ y &= 2x - \pi + \pi \\ y &= 2x\end{aligned}$$

(ii) Gradient of normal at this point $= -\frac{1}{2}$

The equation of the normal will be

$$\begin{aligned}y - \pi &= -\frac{1}{2}\left(x - \frac{\pi}{2}\right) \\ y &= -\frac{1}{2}x + \frac{\pi}{4} + \pi \\ y &= -\frac{1}{2}x + \frac{5}{4}\pi\end{aligned}$$

Question 6

(a) radius

$$\begin{aligned}&= \sqrt{(-2)^2 + 7^2 - 4} \\ &= \sqrt{4 + 49 - 4} \\ &= \sqrt{49} = 7\end{aligned}$$

coordinates of center

$$(-2, 7)$$

(b) sketch (using information above)

(c) Designating Q as the center of the circle, and R the point where the tangent touches the circle

The distance (PQ) from P(6,8) to Q(-2,7) is given by

$$PQ^2 = 8^2 + 1^2 = 65$$

Forming the right-angled triangle PQR

$$\begin{aligned}PR^2 &= PQ^2 - QR^2 \\ &= 65 - 49 \\ &= 16\end{aligned}$$

therefore

$$PR = 4$$

Pure 3 - MAP3, 13 June 2003

Question 1

$$(2+3x)^9 = 2^9 + 2^8.9(3x) + 2^7.\frac{9.8}{2!}(3x)^2 + 2^6.\frac{9.8.7}{3!}(3x)^3..$$

so coefficient of x^3

$$= 2^6.3^3.\frac{9.8.7}{3!}$$
$$= 145152$$

Question 2

(a) $x = 3t - 1 \qquad y = \frac{1}{t}$

$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = -\frac{1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -\frac{1}{t^2}.\frac{1}{3}$$
$$= -\frac{1}{3t^2}$$

(b) at t=1, the gradient of tangent

$$= -\frac{1}{3.1^2} = -\frac{1}{3}$$

so gradient of normal = 3

when $t = 1,\ x = 2,\ y = 1$ - so eqn of normal is

$$y - 1 = 3(x - 2)$$
$$y = 3x - 5$$

Question 3

(a)

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

(b)

$$\frac{dr}{dt} = 3$$

$$\frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt}$$
$$= 2\pi r.3$$

At r = 50

$$= 2\pi.50.3$$
$$= 300\pi$$
$$= 942cm^2/min$$

Question 4

$$N = Ac^t$$

(a) If $N_0 = 1000$

$$1000 = A$$

(b)

$$12000 = 1000c^{60}$$
$$c^{60} = 12$$
$$c = 1.0423$$

(c) (i) Take logs of both sides

$$\ln N = \ln(Ac^t)$$
$$\ln N = \ln A + \ln c^t$$
$$t \ln c = \ln N - \ln A$$
$$t = \frac{\ln N - \ln A}{\ln c}$$

(ii)

$$t = \frac{\ln 1000000 - \ln 1000}{\ln 1.0423}$$
$$= \frac{\ln 1000}{\ln 1.0423}$$
$$= 167mins$$

Question 5

$$f(x) = \frac{1}{3x - 2}$$

(a) (i)

$$f'(x) = -3(3x-2)^{-2}$$
$$f''(x) = 18(3x-2)^{-3}$$

(ii)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$$
$$= -\frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2$$

(b)

$$\frac{(x-1)(x-2)}{x^2(3x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(3x-2)}$$

$$(x-1)(x-2) = Ax(3x-2) + B(3x-2) + Cx^2$$

Let x=0

$$2 = -2B \Rightarrow B = -1$$

Compare coefficients of x

$$-3 = -2A + 3B$$

$$2A = -3 + 3 = 0 \Rightarrow A = 0$$

Compare coefficients of x^2

$$1 = 3A + C$$

$$C = 1$$

(c)

$$\frac{(x-1)(x-2)}{x^2(3x-2)} = -\frac{1}{x^2} + \frac{1}{(3x-2)}$$

now

$$\frac{1}{(3x-2)} = (3x-2)^{-1} = -2^{-1}\left(1 - \frac{3x}{2}\right)^{-1}$$

$$\approx -\frac{1}{2}\left(1 - \left(-\frac{3x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{3x}{2}\right)^2\right)$$

$$\approx -\frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2$$

so

$$\frac{(x-1)(x-2)}{x^2(3x-2)} \approx -\frac{1}{x^2} - \frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2$$

$$\frac{(x-1)(x-2)}{(3x-2)} \approx x^2\left(-\frac{1}{x^2} - \frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2\right)$$

$$\frac{(x-1)(x-2)}{(3x-2)} \approx -1 - \frac{1}{2}x^2 - \frac{3}{4}x^3 - \frac{9}{8}x^4$$

Question 6

(a)

$$dx = \frac{dx}{dt}dt$$

with $\frac{dx}{dt}$ given by formula in question

t	x	$\frac{dx}{dt}$	dt	dx
0	1	1.8	0.3	0.54
0.3	1.54	1.692	0.3	0.5076
0.6	2.0476			

(b)

$$-\int \frac{-dx}{10-x} = \int \frac{dt}{5}$$

$$-\ln(10-x) = \frac{t}{5} + c$$

When $t = 0, x = 1$

$$-\ln(10-1) = 0 + c \Rightarrow c = -\ln 9$$

so eqn becomes

$$-\ln(10-x) = \frac{t}{5} - \ln 9$$

$$t = 5\ln\left(\frac{9}{10-x}\right)$$

when x=2

$$t = 5\ln\frac{9}{8} = 0.589$$

Question 7

(a)

$$\vec{AB} = \begin{bmatrix} 5\text{-}3 \\ 2\text{-}1 \\ 4\text{-}2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 4\text{-}3 \\ \text{-}1\text{-}1 \\ 1\text{-}2 \end{bmatrix} = \begin{bmatrix} 1 \\ \text{-}2 \\ \text{-}1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \mu\begin{bmatrix} 1 \\ \text{-}2 \\ \text{-}1 \end{bmatrix}$$

(c) Taking the vector product of $\vec{AB}$ and $\vec{AC}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$$

Since this is in the exact opposite direction to (-3,-4,5), the latter is perpendicular to the plane.

(d) 2x -3y - z=6 has the perpendicular

$$(2, -3, -1)$$

the angle between the two planes is given by

$$\cos\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$$

(where **a** and **b** are perpendicular to each plane)

$$\cos\theta = \frac{-6 + 12 - 5}{\sqrt{(-3)^2 + (-4)^2 + 5^2}\sqrt{2^2 + (-3)^2 + (-1)^2}}$$

$$\cos\theta = \frac{1}{\sqrt{50}\sqrt{14}}$$

$$\cos\theta = 0.0378$$

$$\theta = 87.8°$$

Pure 4 - MAP4, 13 June 2003

Question 1

(a) (i) the three roots added together

$$\alpha + \beta + (\alpha + \beta) = -2p$$

therefore

$$\alpha + \beta = -p$$

(ii) three roots multiplied together

$$\alpha\beta(\alpha + \beta) = 8$$

$$\alpha\beta(-p) = 8$$

$$\alpha\beta = -\frac{8}{p}$$

(b)

$$\alpha\beta + \alpha(\alpha + \beta) + \beta(\alpha + \beta) = 0$$

$$-\frac{8}{p} + \alpha(-p) + \beta(-p) = 0$$

$$-\frac{8}{p} - p(\alpha + \beta) = 0$$

$$-\frac{8}{p} - p(-p) = 0$$

$$-\frac{8}{p} + p^2 = 0$$

$$p^3 = 8 \Rightarrow p = 2$$

Question 2

(a)

$$4\sinh x + e^x = 5$$

$$4.\frac{e^x - e^{-x}}{2} + e^x = 5$$

$$3e^x - 2e^{-x} = 5$$

multiply through by e^x, and re-arrange

$$3e^{2x} - 5e^x - 2 = 0$$

(b) From above, given equation becomes

$$3e^{2x} - 5e^x - 2 = 0$$

Let $u = e^x$

$$3u^2 - 5u - 2 = 0$$

$$(3u + 1)(u - 2) = 0$$

giving

$u = -\frac{1}{3} \Rightarrow e^x = -\frac{1}{3}$ (invalid because e^x must be +ve)

or

$$u = 2 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$$

Question 3

(a) Argand diagram

Area bounded by circle, center (0,2) with radius of 1

(b) *greatest and least values of argument correspond to arguments of tangents. So can form a right-angled triangle of hypotenuse 2 (along vertical axis), side 1 (a radius), and the other side formed by the tangent to circle from origin.* If α is the angle at the origin

$$\sin\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

The required argument will be therefore

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

and due to symmetry, the argument of the tangent on 'the other side' will be

$$\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$$

Question 4

(a) (i)

$$\int \cosh^2 x \, dx$$

$$= \frac{1}{2}\int (\cosh 2x + 1)dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2x + x\right) + c$$

$$= \frac{1}{4}\sinh 2x + \frac{x}{2} + c$$

(ii)

$$\int x\cosh x \, dx$$

$$= x\sinh x - \int \sinh x \, dx$$

$$= x\sinh x - \cosh x + c$$

(b)

$$x = \cosh t + t$$

$$\frac{dx}{dt} = \sinh t + 1$$

$$\left(\frac{dx}{dt}\right)^2 = \sinh^2 t + 2\sinh t + 1$$

and

$$y = \cosh t - t$$
$$\frac{dy}{dt} = \sinh t - 1$$
$$\left(\frac{dy}{dt}\right)^2 = \sinh^2 t - 2\sinh t + 1$$

therefore

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\sinh^2 t + 2$$
$$= 2(\sinh^2 t + 1) = 2\cosh^2 t$$

(c) (i)

$$S = 2\pi \int_0^1 y\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{1}{2}} dt$$
$$= 2\pi \int_0^1 (\cosh t - t)(2\cosh^2 t)^{\frac{1}{2}} dt$$
$$= 2\pi \int_0^1 (\cosh t - t)\sqrt{2}\cosh t \ dt$$
$$= 2\pi\sqrt{2} \int_0^1 (\cosh t - t)\cosh t \ dt$$

(ii)

$$2\pi\sqrt{2} \int_0^1 (\cosh t - t)\cosh t \ dt$$
$$2\pi\sqrt{2}\left[\int_0^1 \frac{1}{2}(1 + \cosh 2t)dt - \left\{[t\sinh t]_0^1 - \int_0^1 \sinh t dt\right\}\right]$$
$$2\pi\sqrt{2}\left[\frac{t}{2} + \frac{\sinh 2t}{4} - \{t\sinh t - \cosh t\}\right]_0^1$$
$$2\pi\sqrt{2}\left[\frac{t}{2} + \frac{\sinh 2t}{4} - t\sinh t + \cosh t\right]_0^1$$
$$2\pi\sqrt{2}\left\{\left(\frac{1}{2} + \frac{\sinh 2}{4} - \sinh 1 + \cosh 1\right) - (1)\right\}$$
$$2\pi\sqrt{2}\left(\frac{\sinh 2}{4} - \sinh 1 + \cosh 1 - \frac{1}{2}\right)$$

Question 5

(a)

$$\frac{1}{r!} - \frac{1}{(r+1)!}$$
$$= \frac{(r+1)! - r!}{r!(r+1)!}$$
$$= \frac{r!((r+1) - 1)}{r!(r+1)!}$$
$$= \frac{r!(r)}{r!(r+1)!}$$
$$= \frac{r}{(r+1)!}$$

(ii)

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = \sum_{r=1}^{n} \frac{1}{r!} - \sum_{r=1}^{n} \frac{1}{(r+1)!}$$

First term

$$\frac{1}{1} - \frac{1}{2}$$

Second term

$$\frac{1}{2} - \frac{1}{6}$$

It can be seen that adding these two terms together will leave only the first and last term. By extension, for the whole series, only the first and last term will remain, so required sum is

$$\frac{1}{1} - \frac{1}{(n+1)!}$$
$$= 1 - \frac{1}{(n+1)!}$$

(b)

Question 6

(a) (i)

$$|w| = \frac{1}{\sqrt{2}}(-1 + i).\frac{1}{\sqrt{2}}(-1 - i)$$
$$= \frac{1}{2}(1 - i^2) = 1$$

(ii)

$$\theta = \tan^{-1}(-1)$$
$$= \frac{3\pi}{4}$$

so required form is

$$e^{\frac{i3\pi}{4}}$$

(b)

$$z^3 = e^{\frac{i3\pi}{4} \pm i2n\pi}$$
$$z = e^{\frac{i\pi}{4} \pm \frac{i2n\pi}{3}}$$
$$z = e^{\frac{-i5\pi}{12}}, e^{\frac{i\pi}{4}}, e^{\frac{i11\pi}{12}}$$

(c) (i) *(easist to do in polar form)*

$$(1 - w)(1 - w^*)$$
$$= \left(1 - e^{\frac{3\pi i}{4}}\right)\left(1 - e^{\frac{-3\pi i}{4}}\right)$$
$$= 1 + 1 - e^{\frac{3\pi i}{4}} - e^{\frac{-3\pi i}{4}}$$
$$= 2 - 2\cos\frac{3\pi}{4}$$
$$= 2 + \sqrt{2}$$

(ii) The sum of a geometric series where the first term is the "zeroth" term is given by

$$S_n = a\frac{1-r^{n+1}}{1-r}$$

now

$$a = w^0 = 1$$

so

$$\begin{aligned} S_{11} &= \frac{1-w^{12}}{1-w} \\ &= \frac{1-e^{9\pi i}}{1-w} \\ &= \frac{2}{1-w} \end{aligned}$$

multiply top and bottom by $(1+w^*)$, using results above

$$\begin{aligned} &\frac{2(1-w^*)}{2+\sqrt{2}} \\ &= \frac{2\left(1-\frac{1}{\sqrt{2}}(-1-i)\right)}{2+\sqrt{2}} \\ &= \frac{2\left(1-\frac{\sqrt{2}}{2}(-1-i)\right)}{2+\sqrt{2}} \\ &= \frac{2+\sqrt{2}+\sqrt{2}i}{2+\sqrt{2}} \\ &= 1+(\sqrt{2}-1)i \end{aligned}$$

(multiply imaginary part top and bottom by $2-\sqrt{2}$ to get the result on the line above)

Pure 5 - MAP5, 5 June 2003

Question 1

$$\frac{dy}{dx} + y\cot x = 2\cos x$$

I.F =

$$e^{\int \cot x} = e^{\ln \sin x} = e^{\ln \sin x} = \sin x$$

equation becomes

$$\sin x \frac{dy}{dx} + y\cot x \sin x = 2\cos x \sin x$$

$$\sin x \frac{dy}{dx} + y\cos x = 2\cos x \sin x$$

$$\frac{d}{dx}(y\sin x) = 2\sin x\cos x$$

$$y\sin x = \sin^2 x + c$$

$$y = sinx + \frac{c}{\sin x}$$

Alternatively (although not really as elegant)

$$\frac{d}{dx}(y\sin x) = 2\cos x \sin x$$

$$y\sin x = \int \sin 2x\ dx$$

$$y\sin x = -\frac{1}{2}\cos 2x + c$$

$$y = -\frac{1}{2}\left(\frac{\cos 2x}{\sin x}\right) + \frac{c}{\sin x}$$

Question 2

(a) (i)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}...$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}...$$

(ii)

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}..$$

so

$$\ln\cos x = \ln((\cos x - 1) + 1) = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{(-\frac{x^2}{2})^2}{2}$$

above ignores higher order terms

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8}$$

$$= -\frac{x^2}{2} - \frac{x^4}{12}$$

(b) (i)

$$\ln\sec x = -\ln\cos x$$

$$= \frac{x^2}{2} + \frac{x^4}{12}$$

(ii)

$$\lim_{x\to 0}\frac{\ln\sec x}{x^2} = \lim_{x\to 0}\frac{\frac{x^2}{2} + \frac{x^4}{12}}{x^2}$$

$$= \lim_{x\to 0}\left(\frac{1}{2} + \frac{x^2}{12}\right) = \frac{1}{2}$$

Question 3

(a)

$$\int_0^k \frac{x}{(1+x)^2}dx$$

Let u=1+x, du = dx

$$\int_0^k \frac{u-1}{u^2}du = \int_0^k \left(\frac{1}{u} - \frac{1}{u^2}\right)du$$

$$= \left[\ln u + \frac{1}{u}\right]_0^k$$

$$= \left[(\ln(1+x) + \frac{1}{(1+x)})\right]_0^k$$

$$= \left(\ln(1+k) + \frac{1}{1+k}\right) - (1)$$

$$= \ln(1+k) - \frac{k}{1+k}$$

(b) As $k \to \infty$, $\ln(1+k) \to \infty$
and
$\frac{k}{1+k} \to 1$
Therefore the integral does not exist

Question 4

(a) Rejigging the formulae given in the question

$$y'_r = \frac{y_{r+1} - y_{r-1}}{2h}$$

$$y_r{}'' = \frac{y_{r+1} - 2y_r + y_{r-1}}{h^2}$$

Substituting into the differential equation

$$\frac{y_{r+1}-2y_r+y_{r-1}}{h^2}+\frac{4(y_{r+1}-y_{r-1})}{2h}+2x_ry_r=0$$

$$y_{r+1}-2y_r+y_{r-1}+2h(y_{r+1}-y_{r-1})+2x_ry_rh^2=0$$

$$y_{r+1}(1+2h)=(2-2h^2x_r)y_r+(2h-1)y_{r-1}$$

$$y_{r+1}=\frac{2(1-h^2x_r)y_r+(2h-1)y_{r-1}}{(1+2h)}$$

(b)

r	x_r	y_r	y_{r+1}
0	0	1	
1	0.1	1.2	1.331

Question 5

(a) if

$$y=Ax^2e^{-2x}$$

then

$$y'=A(-2x^2e^{-2x}+2xe^{-2x})$$

$$y'=2Ae^{-2x}(x-x^2)$$

and

$$y''=2A\left(e^{-2x}(1-2x)+(-2e^{-2x}(x-x^2)\right)$$

$$=2Ae^{-2x}\left[1-2x+(x-x^2)(-2)\right]$$

$$=2Ae^{-2x}\left[2x^2-4x+1\right]$$

substitute into equation

$$2Ae^{-2x}(2x^2-4x+1)+8Ae^{-2x}(x-x^2)+4Ae^{-2x}(x^2)=6e^{-2x}$$

$$2A(2x^2-4x+1)+8A(x-x^2)+4Ax^2=6$$

comparing coefficients

$$2A=6\Rightarrow A=3$$

(b) Auxiliary Eqn

$$m^2+4m+4=0$$

$$(m+2)(m+2)=0$$

giving solution

$$(A+Bx)e^{-2x}$$

so general solution is

$$y=(A+Bx)e^{-2x}+3x^2e^{-2x}$$

$$=e^{-2x}\left(A+Bx+3x^2\right)$$

(c) Since

$$x^ne^{-kx}\to 0\ as\ x\to\infty$$

then

$$\lim_{x\to\infty}y=0$$

Question 6

(a) (i)

$$x^2+y^2=16$$

$$(r\cos\theta)^2+(r\sin\theta)^2=16$$

$$r^2(\cos^2\theta+\sin^2\theta)=16$$

$$r^2=16\Rightarrow r=4$$

(ii)

$$xy=4$$

$$(r^2\cos\theta\sin\theta)=4$$

$$r^2.\frac{1}{2}.\sin 2x=4$$

$$r^2=8cosec2\theta$$

(b) At intersection

$$16=8cosec2\theta$$

$$cosec2\theta=2$$

$$\sin 2\theta=\frac{1}{2}$$

giving for the principal value (at A)

$$2\theta=\frac{\pi}{6}\Rightarrow\theta=\frac{\pi}{12}$$

and for the other value (at B)

$$2\theta=\frac{5\pi}{6}\Rightarrow\frac{5\pi}{12}$$

(c) Area enclosed between C_1 and C_2 is given by

$$\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}}\frac{1}{2}.4^2\ d\theta-\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}}\frac{1}{2}8cosec2\theta\ d\theta$$

(using Area $=\frac{1}{2}\int r^2\ d\theta$*)*

$$=\left[8\theta+\frac{4}{2}\ln(cosec2\theta+\cot 2\theta\right]_{\frac{\pi}{12}}^{\frac{5\pi}{12}}$$

$$=\left(\frac{40\pi}{12}+\frac{4}{2}\ln(cosec\frac{10\pi}{12}\theta+\cot\frac{10\pi}{12}\right)-$$

$$\left(\frac{8\pi}{12}+\frac{4}{2}\ln(cosec\frac{2\pi}{12}+\cot\frac{2\pi}{12}\right)$$

$$=\left(\frac{10\pi}{3}+2\ln(cosec\frac{5\pi}{6}\theta+\cot\frac{5\pi}{6}\right)-$$

$$\left(\frac{2\pi}{3}+2\ln(cosec\frac{\pi}{6}+\cot\frac{\pi}{6}\right)$$

$$=\left(\frac{10\pi}{3}+2\ln(2-\sqrt{3}\right)-\left(\frac{2\pi}{3}+2\ln(2+\sqrt{3}\right)$$

$$= \frac{8\pi}{3} + 2\ln\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)$$

$$= \frac{8\pi}{3} - 2\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

$$= \frac{8\pi}{3} - 2\ln\left(\frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}\right)$$

$$= \frac{8}{3} - 4\ln(2+\sqrt{3})$$

Pure 6 - MAP6, 5 June 2003

Question 1

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{-2} = k$$

where k is a constant, so

$$x = k,\ y = -k,\ z = -2k$$

so using the given transformation on a vector formed from the above, gives

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} k \\ -k \\ -2k \end{bmatrix} = \begin{bmatrix} -k \\ k \\ 2k \end{bmatrix} = - \begin{bmatrix} k \\ -k \\ -2k \end{bmatrix}$$

Question 2

(a) A matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

describes a counter-clockwise rotation about the origin, and this will fit the bill if $\theta = \frac{1}{3}\theta$, producing

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(b) Since **M** represents the angle $\frac{\pi}{3}$, it will neeed six such transformations to achieve a a total rotation of 2π, where $\cos\theta = 1$ and $\sin\theta = 0$ i.e.

$$\mathbf{M}^6 = \mathbf{I}$$

Question 3

(a) (i)

$$\begin{vmatrix} 2 & 1 & -1 \\ k & 3 & k \\ 3 & 2 & -3 \end{vmatrix}$$

$$= 2(-9 - 2k) - (-3k - 3k) + (-1)(2k - 9)$$

$$-18 - 4k + 3k + 3k - 2k + 9 = -9$$

i.e. determinant is independent of k.

(ii) The fact that the determinant is not zero, means that the equations have a unique solution.

(b) (i) if $k = 1$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ 3 & 2 & -3 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -11 & 6 & -7 \\ 1 & -3 & -1 \\ 4 & -3 & 5 \end{bmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} -11 & 1 & 4 \\ 6 & -3 & -3 \\ -7 & -1 & 5 \end{bmatrix}$$

Since the determinant of **A** is -9, then

$$\mathbf{A}^{-1} = -\frac{1}{9} \begin{bmatrix} -11 & 1 & 4 \\ 6 & -3 & -3 \\ -7 & -1 & 5 \end{bmatrix}$$

(ii) Expressing the system of equations in matrix form

$$\mathbf{Ax} = \mathbf{y}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} -11 & 1 & 4 \\ 6 & -3 & -3 \\ -7 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Question 4

(a) (i) Characteristic equation

$$\begin{vmatrix} 5 - \lambda & -2 \\ 12 & -5 - \lambda \end{vmatrix} = 0$$

$$(5 - \lambda)(-5 - \lambda) - (12)(-2) = 0$$

$$-25 - 5\lambda + 5\lambda + \lambda^2 + 24 = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda = 1 \ or \ -1$$

When $\lambda = 1$

$$\begin{bmatrix} 4 & -2 \\ 12 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

giving

$$4x - 2y = 0 \Rightarrow 2x - y = 0$$

therefore eigenvector is

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

When $\lambda = -1$

$$\begin{bmatrix} 6 & -2 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

giving

$$6x - 2y = 0 \Rightarrow 3x - y = 0$$

therefore eigenvector is

$$\lambda \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(ii)

$$\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(iii)

$$\mathbf{D}^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) (i) In general

$$\mathbf{M}^{\mathrm{n}} = \mathbf{UDU}^{-1}\mathbf{UDU}^{-1}\mathbf{UDU}^{-1}...(\text{ntimes})$$

Since

$$\mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$$

then

$$\mathbf{M}^{\mathrm{n}} = \mathbf{UD}^{\mathrm{n}}\mathbf{U}^{-1}$$

(ii)

$$\mathbf{M}^{10} = \mathbf{UD}^{10}\mathbf{U}^{-1} = \mathbf{I}$$

$$\mathbf{M}^{11} = \mathbf{UD}^{11}\mathbf{U}^{-1} = \mathbf{UDU}^{-1} = \mathbf{M}$$

Question 5

(a) (i) The equation tells us that $\mathbf{r} - \mathbf{a}$ is parallel to $\mathbf{b}$, ie.

$$\mathbf{r} - \mathbf{a} = \lambda\mathbf{b}$$

$$\mathbf{r} = \lambda\mathbf{b} + \mathbf{a}$$

i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 10 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 + 2\lambda \\ 10 - 3\lambda \\ -4 + \lambda \end{bmatrix}$$

(ii) Need to find the point where $\mathbf{r}$ is perpendicular to $\mathbf{b}$, i.e where

$$\begin{bmatrix} -4 + 2\lambda \\ 10 - 3\lambda \\ -4 + \lambda \end{bmatrix} . \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 0$$

$$-8 + 4\lambda - 30 + 9\lambda - 4 + \lambda = 0$$

$$14\lambda - 42 \Rightarrow \lambda = 3$$

giving the coordinates of N as

$$(2, 1, -1)$$

(b) Need to find line perpendicular to both $\mathbf{b}(2, -3, 1)$ and vector along ON (2,1,-1),ie.

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

By analogy with the equation for the line l, $\mathbf{c}$ will be a vector to the point N, i.e.

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

and $\mathbf{d}$ will be a line parallel to the line we are seeking, i.e.

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \textit{or any multiple of this vector}$$

Question 6

(a)

$$\vec{AD} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

using this to extrapolate from B gives the coordinates of E, i.e.

$$(2, 1, p + 5)$$

(b) (i)

$$\vec{AB} = \begin{bmatrix} -2 \\ 0 \\ p \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

therefore

$$\vec{AB} \times \vec{AC} = \begin{bmatrix} -2 \\ 0 \\ p \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2p \\ 4 + 2p \\ 4 \end{bmatrix}$$

(ii)

$$\vec{AD}.\left(\vec{AB} \times \vec{AC}\right) = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} . \begin{bmatrix} 2p \\ 4 + 2p \\ 4 \end{bmatrix}$$

$$= 6p - 4 - 2p + 20$$

$$= 4p + 16$$

(c) (i) Volume of prism =

$$\frac{1}{2}\left(\vec{AD}.\left(\vec{AB} \times \vec{AC}\right)\right)$$

if p =1, Volume = $\frac{1}{2}(4 + 16) = 10$

(ii) When p=-4

$$\vec{AD}.\left(\vec{AB} \times \vec{AC}\right) = -16 + 16 = 0$$

therefore

$$\vec{AD}, \vec{AB}, \text{ and } \vec{AC} \text{ are coplanar}$$

Mechanics 1 - MAM1, 22 May 2003

Question 1

(a)

$$v^2 = 2as$$

since u = 0; so

$$v^2 = 2 \times 9.8 \times 10 = 196$$

$$v = 14ms^{-1}$$

(b) air resistance is negligible
book modeled as a particle

Question 2

(a) (i) The acceleration is given by the gradient of the line

$$a = \frac{5U - 3U}{3} = \frac{2U}{3}$$

(ii) The distance traveled is represented by the area under the graph *(Could treat area as two shapes, or as a trapezium (as here))*

$$= (4U \times 3) = 12U$$

(b) Equate distance traveled to area under relevant triangle

$$20 = \frac{1}{2}(4) \times 5U$$

giving

$$U = 2$$

Question 3

(a) diagram showing W vertically, R in opposite direction and the friction F = μR acting horizontally to the left, plus forces shown in question.

(b) Resolving horizontally

$$F = 12 + 15 \cos 30$$

$$= 12 + 15\frac{\sqrt{3}}{2}$$

$$= 12 + 12.990$$

$$= 25.0N$$

(c) Since the system is in limiting equilibrium

$$F = \mu R$$

$$R = \frac{F}{\mu} = \frac{25}{\frac{1}{3}} = 75N$$

(d) Resolving vertically

$$W = R + 15 \sin 30$$

$$W = 75 + 7.5 = 82.5N$$

Question 4

(a)

$$\mathbf{v} = (2\text{t} - 6)\mathbf{i} + \text{t}^2\mathbf{j}$$

(b) When Q is moving parallel to **j**

$$2t - 6 = 0 \Rightarrow t = 3$$

(c)

$$\mathbf{a} = 2\mathbf{i} + 2\text{t}\mathbf{j}$$

this acceleration will not be constant because it is a function of t

Question 5

(a) Difference of position vectors

$$\begin{bmatrix} 2 \\ 8 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Dividing by 3 gives

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ms}^{-1}$$

(b) Using Conservation of Momentum
If velocity of R is **v**

$$0.2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} -5 \\ 5 \end{bmatrix} = (0.2 + 0.1) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix} = 0.3 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

so

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(c) Position vector after 3 secs

$$= \begin{bmatrix} 2 \\ 8 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 17 \end{bmatrix}$$

Question 6

(a) (i) diagram of trailer, showing Weight 250g, Reaction, Friction 100N to left, and force in towbar (T, say) to right.

Taking the left direction as positive

$$R - T = ma$$

$$T = R - ma$$
$$= 100 - (250 \times 0.5)$$
$$= 100 - 125 = -25N$$

i.e. 25N to the right

(ii) diagram with 25N to the right, and 500 N to the left, as well as the braking force (B, say) to the left

Taking the left as the positive direction

$$B - 25 + 500 = ma$$

$$B = ma - 475$$
$$B = 1250 \times 0.5 - 475 = 150N$$

(b) For trailer, resolving horizontally

$$T = 100N$$

Question 7

(a)

$$u_H = V \sin\alpha = 21(0.7) = 14.7ms^{-1}$$

so

$$v_H = u_H - gt$$
$$t = \frac{u_H - v_H}{g}$$
$$= \frac{14.7 - 0}{9.8} = 1.5s$$

(b) (i)

$$v_H^2 = u_H^2 - 2gs$$

$$s = \frac{u^2}{2g} = \frac{14.7^2}{2 \times 9.8}$$
$$= 11.025m$$

so distance between this and top of the tree = $3.025m$ = 3.03m to 3 sig. figs.

(ii)

$$s = ut - \frac{1}{2}gt^2$$

$$8 = 21 \times 0.7t - \frac{1}{2} \times 9.8 \times t^2$$

$$4.9t^2 - 14.7t + 8 = 0$$
$$t = \frac{14.7 \pm \sqrt{14.7^2 - 4 \times 4.9 \times 8}}{9.8}$$
$$= \frac{14.7 \pm \sqrt{59.29}}{9.8}$$
$$= \frac{14.7 \pm 7.7}{9.8}$$
$$t = 0.71s \text{ to first reach 8m}$$

(and also gives 2.286 s for the second solution)

(iii) Subtract one solution for t above from the other gives

$$2.286 - 0.714 = 1.57 \; to \; 3 \; sig \; figs$$

Mechanics 2 - MAM2, 17 June 2003

Question 1

(a)

$$\omega = \frac{2\pi}{27.32 \times 24 \times 60 \times 60}$$

$$= 2.66 \times 10^{-6} rad\ s^{-1}$$

(b)

$$v = r\omega$$

$$= 3.844 \times 10^8 \times 2.66 \times 10^{-6}$$

$$= 10.22 \times 10^2 = 1022 ms^{-1}$$

Question 2

(a) symmetry

(b)

$$M\bar{x} = 96m \times 4 + 160m \times 10$$

where M is the total mass

$$M\bar{x} = 1984m$$

$$\bar{x} = \frac{1984m}{256m} = 7.75m$$

(c) By symmetry, the Center of Mass must lie 7.75m from AR

So construct a right-angled triangle with apex B, a vertical line of unknown length from B (passing through the center of mass), the other line from B being 12.5 cm long, and the other side being 7.5 cm.

Therefore

$$\tan\theta = \frac{7.75}{12.5}$$

$$\theta = 32^\circ$$

Question 3

(a) (i) Separation Speed = e . (Approach Speed)

$$v_B - v_A = \frac{1}{2}(u_A)$$

$$2 - v_A = 2 \Rightarrow v_A = 0$$

(ii) Using conservation of momentum

$$m_2 v_B = m_1 u_A$$

$$m_2(2) = m_1(4)$$

$$\frac{m_1}{m_2} = \frac{1}{2}$$

or as ratio

$$m_1 : m_2 = 1 : 2$$

(iii) KE before

$$= \frac{1}{2} m_1 .16 = 8m_1$$

KE after

$$= \frac{1}{2} m_2 4 = 2m_2$$

Percentage of KE remaining after

$$= \frac{2m_2}{8m_1} \times 100 = 50\%$$

Therefore percentage loss = 50%

(b)

$$v_B - v_A = e u_A$$

$$2 - v_A = e(4)$$

$$v_A = 2 - 4e$$

For the two bodies to be moving in opposing directions we require v_A to be negative, i.e.

$$4e > 2$$

$$e > \frac{2}{4} = \frac{1}{2}$$

and by definition $e \leq 1$. So

$$\frac{1}{2} < e \leq 1$$

Question 4

(a) Need to find speed at 14 meters

$$v^2 = u^2 + 2gs$$

$$v^2 = 2(9.8)(14)$$

$$v^2 = 16.6 ms^{-1}$$

(b) (i) PE lost = Elastic Energy Gained

$$mgh = \frac{1}{2}kx^2$$

$$70(9.8)(14+x) = \frac{1}{2}(196)x^2$$

$$980 + 70x = 10x^2$$

$$x^2 - 7x - 98 = 0$$

(ii) above equation factorizes to

$$(x-14)(x+7) = 0$$

so

$$x = -7, \ or \ 14$$

giving the answer x = 14 m

(iii) Force upwards due to spring

$$= kx = 196 \times 14 = 2744$$

with downwards treated as positive, and using

$$F = ma$$

$$70(9.8) - 2744 = 70a$$

$$a = -29.4ms^{-2}$$

Question 5

(a)

$$\mathbf{a} = \begin{bmatrix} -4e^{2t} \\ 2e^t \end{bmatrix}$$

$$\mathbf{F} = 4\begin{bmatrix} -4e^{2t} \\ 2e^t \end{bmatrix} = \begin{bmatrix} -16e^{2t} \\ 8e^t \end{bmatrix}$$

(b) (i)

$$\mathbf{F}.\mathbf{v} = 4\begin{bmatrix} -4e^{2t} \\ 2e^t \end{bmatrix}.\begin{bmatrix} 1-2e^{2t} \\ 2e^t \end{bmatrix}$$

$$= 4\left[(-4e^{2t})(1-2e^{2t}) + 2e^t.2e^t\right]$$

$$4\left[-4e^{2t} + 8e^{4t} + 4e^{2t}\right]$$

$$32e^{4t}$$

(ii) Work done

$$= \int_0^{\ln 3} 32e^{4t}.dt$$

$$\left[8e^{4t}\right]_0^{\ln 3}$$

$$8e^{4\ln 3} - 8$$

$$8e^{\ln\left(3^4\right)} - 8$$

$$8.3^4 - 8$$

$$640J$$

Question 6

(a) At C, weight is the only force on car, so

$$mg = \frac{mv^2}{r}$$

giving

$$v^2 = rg$$

(b) K. E. at B = K. E. at C + PE gained from B to C

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg(2r)$$

$$\frac{1}{2}mu^2 = \frac{1}{2}mrg + 2mgr$$

$$u^2 = 5gr$$

(c) Between A and B,
PE lost = KE gained

$$mgh = \frac{1}{2}mu^2$$

$$mgh = \frac{1}{2}m(5gr)$$

$$h = \frac{5r}{2}$$

so

$$k = 2.5$$

(d) Include friction in the formulation, or other resistive forces

Mechanics 3 - MAM3, 23 June 2003

Question 1

(a)

$$I = mr^2$$

$$= (0.4)(0.05)^2$$

$$= 0.001\ kgm^2$$

(b)

$$\text{Moment of Force} = \text{I}\ddot{\theta}$$

$$10 \times 0.0025 = 0.001 \times \ddot{\theta}$$

$$\ddot{\theta} = \frac{0.025}{0.001} = 25rad\ s^{-1}$$

(c)

$$\omega = \ddot{\theta}t = 25 \times 3 = 75rad\ s^{-1}$$

Question 2

(a) Diagram showing Vertical Forces of Weight of Ladder(20g), Weight of Man (80g) and Reaction at B (R), and Horizontal Forces of Friction at B (0.4R), Reaction at A (S),

(b) (i) Resolving vertically

$$100g = R$$

Resolving horizontally

$$S = 0.4R = 0.4(100g) = 40 \times 9.8 = 392N$$

(ii) Taking moments about B

$$S(5\sin 60) = 80g(x\cos 60) + 20g(2.5\cos 60)$$

$$\frac{5\sqrt{3}}{2}(392) = 40gx + 25g$$

$$980\sqrt{3} = 392x + 245$$

$$x = \frac{980\sqrt{3} - 245}{392} = 3.71m$$

Question 3

(a) Moment of Inertia of disk about O

$$= \frac{ma^2}{2}$$

Using the Parallel Axes Theorem, the M of I about A

$$= \frac{ma^2}{2} + m(2a)^2$$

$$= \frac{ma^2}{2} + 4ma^2$$

$$= \frac{9}{2}ma^2$$

(b) (i) PE lost

$$= mg(2a\cos\theta)$$

KE gained

$$\frac{1}{2}I\dot{\theta}^2$$

$$= \frac{9}{4}ma^2\dot{\theta}^2$$

Equating

$$2ga\cos\theta = \frac{9}{4}a^2\dot{\theta}^2$$

$$\dot{\theta}^2 = \frac{8g\cos\theta}{9a}$$

(ii) Let required component be R

$$R - mg\cos\theta = mr\dot{\theta}^2$$

$$R = m(2a)\frac{8g\cos\theta}{9a} + mg\cos\theta$$

$$R = \frac{16}{9}mg\cos\theta + mg\cos\theta$$

$$R = \frac{25}{9}mg\cos\theta$$

Question 4

(a) Each internal angle is 120°

Resolving vertically (taking upwards as positive)

$$2\cos 30 - 1\cos 30 + 2\cos 30 - 3\cos 30 = 0$$

Resolving horizontally (taking forces to the right as positive)

$$1 - 3\sin 30 - 2\sin 30 + 3 - 1\sin 30 - 2\sin 30$$

$$4 + \sin 30(-3 - 2 - 1 - 2)$$

$$4 + \frac{1}{2}(-8) = 0$$

(b) (i) Taking moments counter-clockwise about B

$$-3a\cos 30 - 3(2a\cos 30) + 2(2a\cos 30) + 1(a\cos 30)$$
$$= -3a\cos 30 - 6a\cos 30 + 4a\cos 30 + \cos 30$$
$$= -4a\cos 30$$
$$= -2\sqrt{3}a$$

(ii) We have used the standard convention that counter-clockwise is positive, so above answer shows that the sense of the couple L is clockwise.

Question 5

(a) Three forces are acting - The reaction R, the friction F up the plane and parallel to it, and the weight W thru the center of gravity

(b) Center of Gravity of a cone is $\frac{1}{4}\times$ height from the base, ie.
$$\frac{1}{4} \times 6a = \frac{3a}{2}$$
Forming a right-angled triangle with the apex at the center of gravity and a horizontal line at the base representing the radius of the cone. The sides straddling the right angle will be of length a and $\frac{3a}{2}$. Calling the angle at the apex ϕ, then we know that the cone will topple when the apex of the triangle lies to the left the left-hand corner, i.e. when the line of action of the weight W lies to the left of the left-hand corner. This will be when the angle of incline θ is $> \phi$, ie. when

$$\tan\theta > \frac{a}{\frac{3a}{2}} \Rightarrow \tan\theta > \frac{2}{3}$$

$$\theta > \tan^{-1}\frac{2}{3} \Rightarrow \theta > 33.7^\circ$$

(c) Resolving
$$R = W\cos\theta$$
$$W\sin\theta = F$$
cone will not slide when $F \leq \mu R$
$$W\sin\theta \leq 0.6W\cos\theta$$
$$\tan\theta \leq 0.6$$
$$\theta \leq 31.0^\circ$$
Therefore the cone will slide before it topples

Question 6

(a) gain in PE by the 1kg mass
$$= gx\ J$$
loss in PE by the 1.5kg mass
$$= 1.5gx\ J$$
therefore total loss in PE
$$= 0.5gx\ J$$

(b) Total KE of masses
$$= \frac{1}{2}(1)v^2 + \frac{1}{2}(1.5)v^2$$
$$= 1.25v^2\ J$$
KE of disk
$$= \frac{1}{2}I\omega^2$$
Now
$$\omega = \frac{v}{r} = 4v$$
so, KE of disk
$$= \frac{1}{2}(0.25)(4v)^2$$
$$= 2v^2\ J$$
Therefore, the total KE
$$= 3.25v^2\ J$$

(c) Equating gain in KE to loss of PE
$$3.25v^2 = 0.5gx$$
Differentiate by t
$$3.25(2v)\frac{dv}{dt} = 0.5g\frac{dx}{dt}$$
where $\frac{dx}{dt} = v$, so
$$\frac{dv}{dt} = \frac{0.5g}{6.5} = 0.754ms^{-2}$$

Mechanics 4 - MAM4, 25 June 2003

Question 1

(a) Using equation of motion

$$0.2a = -0.2g - 0.002v^2$$

$$a = -g - 0.01v^2$$

$$v\frac{dv}{dx} = -(9.8 + 0.01v^2)$$

Note : using

$$a = v\frac{dv}{dx}$$

from

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

(b) Rejigging above equation

$$-\frac{v}{9.8 + 0.01v^2}dv = dx$$

$$-50\int_7^0 \frac{0.02v}{9.8 + 0.01v^2}dv = \int_0^h dx$$

$$[x]_0^h = -50\left[\ln(9.8 + 0.01v^2)\right]_7^0$$

$$h = -50\left(\ln\frac{9.8}{10.29}\right)$$

$$= 2.44m$$

Question 2

(a)

$$x + y + z = k$$

where k is a constant. Therefore

$$\dot{x} + \dot{y} + \dot{z} = 0$$

$$\dot{y} = -\dot{x} - \dot{z}$$

$$\dot{y} = 3x - 2y$$

(b) (i)

$$\dot{x} = -3x$$

$$\int\frac{dx}{x} = -\int 3dt$$

$$\ln x = -3t + C$$

when $t = 0, x = 4$

$$\ln 4 = C$$

so

$$\ln x = -3t + \ln 4$$

$$x = e^{-3t+\ln 4}$$

$$x = e^{-3t}e^{\ln 4}$$

$$x = 4e^{-3t}$$

inserting this into expression from (a)

$$\dot{y} = 3\left(4e^{-3t}\right) - 2y$$

$$\dot{y} + 2y = 12e^{-3t}$$

(ii) I.F.

$$= \int e^{2t}dt = \frac{1}{2}e^{2t}$$

eqn becomes

$$\dot{y}\left(\frac{1}{2}e^{2t}\right) + 2y\left(\frac{1}{2}e^{2t}\right) = 12e^{-3t}\left(\frac{1}{2}e^{2t}\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}ye^{2t}\right) = 6e^{-t}$$

$$\frac{1}{2}ye^{2t} = -6e^{-t} + C$$

When t=0, y=0

$$0 = -6 + C \Rightarrow C = 6$$

so expression becomes

$$\frac{1}{2}ye^{2t} = -6e^{-t} + 6$$

$$y = -12e^{-3t} + 12e^{-2t}$$

(iii) Differentiating y

$$\dot{y} = 36e^{-3t} - 24e^{-2t}$$

This is a maximum when

$$36e^{-3t} - 24e^{-2t} = 0$$

$$36e^{-t} - 24 = 0$$

$$e^{-t} = \frac{2}{3}$$

$$-t = \ln\frac{2}{3} = -\ln\frac{3}{2}$$

$$i.e. \quad t = \ln 1.5$$

inserting this into expression for y from (ii)

$$y = -12e^{-3\ln 1.5} + 12e^{-2\ln 1.5}$$

$$y = 1.78\ g$$

Question 3

(a)

$$m = 1000kg$$
$$r = 20kgs^{-1}$$
$$u = 800ms^{-1}$$

Consider time t
Velocity of car = v
Mass of car = m-rt

At time $t + \delta t$
Velocity of car = $v + \delta v$
Mass of car = $m - rt - r\delta t$

At time $t + \delta t$
Velocity of fuel ejected since time $t = v - u$
Mass of this fuel = $r\delta t$

Calculating change in momentum and equating to the impulse

$$(m-rt-r\delta t)(v+\delta v)+(r\delta t)(v-u)-(m-rt)v = -0.1v\delta t$$

$$mv-vrt-vr\delta t+m\delta v-rt\delta v-r\delta t\delta v+vr\delta t-ur\delta t-vm+vrt = -0.1v\delta t$$

$$m\delta v - rt\delta v - r\delta t\delta v - ur\delta t = -0.1v\delta t$$

taking only terms to first order and dividing by δt

$$\frac{\delta v}{\delta t}(m - rt) = ur - 0.1v$$

As $\delta t \to 0$

$$\frac{dv}{dt} = \frac{ur - 0.1v}{m - rt}$$

(b)

$$\frac{dv}{dt} = \frac{ur - 0.1v}{m - rt}$$

$$\int_0^v \frac{dv}{ur - 0.1v} = \int_0^{10} \frac{dt}{m - rt}$$

$$\int_0^v \frac{dv}{16000 - 0.1v} = \int_0^{10} \frac{dt}{1000 - 20t}$$

$$-10\left[\ln(16000 - 0.1v)\right]_0^v = -\frac{1}{20}\left[\ln(1000 - 20t)\right]_0^{10}$$

$$-10\left(\ln(16000 - 0.1v) - \ln(16000)\right) = -\frac{1}{20}\left(\ln 800 - \ln 1000\right)$$

$$-10\left(\ln(16000 - 0.1v) - \ln(16000)\right) = -\frac{1}{20}\left(\ln 0.8\right)$$

$$-10\left(\ln(16000 - 0.1v)\right) = -\frac{1}{20}\left(\ln 0.8\right) - 10\ln(16000)$$

$$-10\left(\ln(16000 - 0.1v)\right) = 96.79228$$

$$\ln(16000 - 0.1v) = 9.679228$$

$$16000 - 0.1v = e^{9.679228}$$

$$v = 10(16000 - e^{9.679228})$$

$$v = 178ms^{-1}$$

Question 4

(a) acceleration of missile towards Earth

$$m_3 a = -\frac{Gm_1m_3}{x^2} - \left(-\frac{Gm_2m_3}{(d-x)^2}\right)$$

$$a = \frac{Gm_2}{(d-x)^2} - \frac{Gm_1}{x^2}$$

(b)

$$0 = \frac{G(0.0123m_1)}{(d-x)^2} - \frac{Gm_1}{x^2}$$

$$\frac{1}{x^2} = \frac{0.0123}{(d-x)^2}$$

$$\frac{(d-x)^2}{x^2} = 0.0123$$

$$\frac{d-x}{x} = \sqrt{0.0123}$$

$$\frac{d}{x} - 1 = \sqrt{0.0123}$$

$$\frac{d}{x} = \sqrt{0.0123} + 1$$

therefore

$$x = \frac{1}{\sqrt{0.0123} + 1}\, d$$

$$x = 0.90d$$

(Since $x < d$, we have just considered the positive square root)

(c) from(a)

$$v\frac{dv}{dx} = \frac{Gm_2}{(d-x)^2} - \frac{Gm_1}{x^2}$$

$$\int_u^v v\,dv = \int_R^x \frac{Gm_2}{(d-x)^2}dx - \int_R^x \frac{Gm_1}{x^2}dx$$

$$\left[\frac{1}{2}v^2\right]_u^v = \left[\frac{Gm_2}{(d-x)} + \frac{Gm_1}{x}\right]_R^x$$

$$\frac{1}{2}v^2 - \frac{1}{2}u^2 = \frac{Gm_2}{(d-x)} - \frac{Gm_2}{(d-R)} + \frac{Gm_1}{x} - \frac{Gm_1}{R}$$

$$v = \sqrt{2Gm_2\left(\frac{1}{d-x} - \frac{1}{d-R}\right) + 2Gm_1\left(\frac{1}{x} - \frac{1}{R}\right) + u^2}$$

Question 5

(a) Auxiliary Equation

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$m = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$m = -1 \pm 2i$$

So Complementary Function is

$$\theta = e^{-t}(A\cos 2t + B\sin 2t)$$

To find Particular Integral

Let trial function be $\theta_0 = k$, so $\dot{\theta_0} = 0$
Therefore

$$5k = 0.3$$

$$k = 0.06$$

General Solution is

$$\theta = e^{-t}(A\cos 2t + B\sin 2t) + 0.06$$

(b) When t=0, $\theta = 0.2$

$$0.2 = A + 0.06$$

$$A = 0.14$$

Differentiating GS

$$\dot{\theta} = e^{-t}(-2A\sin 2t + 2B\cos 2t) - e^{-t}(A\cos 2t + B\sin 2t)$$

When t=0, $\dot{\theta}$ =0

$$0 = 2B - A$$

$$B = \frac{1}{2}A = \frac{1}{2}(0.14) = 0.07$$

so required solution is

$$\theta = e^{-t}(0.14\cos 2t + 0.07\sin 2t) + 0.06$$

(c) As $t \to \infty$

$$e^{-t} \to 0$$

so

$$e^{-t}(0.14\cos 2t + 0.07\sin 2t) \to 0$$

so as $t \to \infty$

$$\theta \to 0.06$$