AQA Board. **GCE Maths (Specification A).**

Model Answers

June 2002

Note comment on paper about stating final answers to 3 significant figures, in general

Pure Maths 1 - MAP1, 28 May 2002

Question 1

(a) Consider

$$x^4 - 5 + 2x = 0$$

When x=1.2, LHS of above = -0.5264

When x=1.3, LHS of above = 0.4561

Since the value of LHS changes sign, there must be a root between 1.2 and 1.3

(b) When x= 1.25, LHS = -0.05859375

root therefore appears somewhere between 1.25 and 1.3

Question 2

(a)

$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$= 8(4 + (15)3)$$
$$= 392$$

(b) (i)

$$u_1 = 50 - 3 = 47$$
$$u_2 = 50 - 6 = 44$$
$$u_3 = 50 - 9 = 41$$
$$u_4 = 50 - 12 = 38$$

(ii)

$$u_{16} = 50 - 48 = 2$$

and common difference is -3

so only 16 positive terms

Question 3

(a) (i)

$$y = 2x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = x^{-\frac{1}{2}}$$

(ii)

$$y = \ln(x+1) \Rightarrow \frac{dy}{dx} = \frac{1}{x+1}$$

(b)

$$\int_1^4 \left(x^{-\frac{1}{2}} + \frac{1}{x+1}\right) dx$$
$$\left[2x^{\frac{1}{2}} + \ln(x+1)\right]_1^4$$
$$(4 + \ln 5) - (2 + \ln 2)$$
$$2 + \ln 5 - \ln 2$$
$$2 + \ln\frac{5}{2}$$

Question 4

(a)

$$y = e^{-2x}$$
$$\frac{dy}{dx} = -2e^{-2x}$$
$$\frac{d^2y}{dx^2} = 4e^{-2x}$$

(b) (i)

$$\int e^{-2x}\,dx$$
$$= -\frac{e^{-2x}}{2} + c$$

(ii) Integral between x=0 and x=1 would produce

$$= -\left[\frac{e^{-2x}}{2}\right]_0^1$$
$$= -\left\{\left(\frac{e^{-2}}{2}\right) - \left(\frac{1}{2}\right)\right\}$$
$$\frac{1}{2} - \frac{1}{2e^2}$$
$$= \frac{e^2 - 1}{2e^2}$$

Question 5

(a) (i)

$$\frac{1}{\sqrt{2}}$$

(ii)

$$\frac{\sqrt{3}}{2}$$

(iii)

$$\sqrt{3}$$

(b)

$$\sin^2 x = \frac{1}{2}$$
$$\sin x = \frac{1}{\sqrt{2}}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4} \quad \text{between} \quad 0 \quad \text{and} \quad \pi$$

(c) $\sin^2 x > \frac{1}{2}$ when

$$\frac{\pi}{4} < x < \frac{3\pi}{4}$$

(d)

$$\sin^2 x > \frac{1}{2}$$
$$1 - \cos^2 x > \frac{1}{2}$$
$$\frac{1}{2} > \cos^2 x$$

Question 6

(a) (i)

$$\alpha = \frac{\pi}{3}$$

(ii) With AC=AB = r, length of arc BC

$$= r\alpha$$
$$= 6.\frac{\pi}{3}$$
$$= 2\pi cm$$

(iii) Area of $\triangle ABC$

$$= \frac{1}{2}AB.AC \sin\alpha$$
$$= \frac{1}{2}.6.6.\frac{\sqrt{3}}{2}$$
$$= 9\sqrt{3}$$

(iv) Area of sector ABC

$$= \frac{1}{2}r^2\alpha$$
$$= \frac{1}{2}.6^2.\frac{\pi}{3}$$
$$= 6\pi cm^2$$

(b) (i) Boundary is three times the answer from (a)(ii), i.e.

$$6\pi = 19cm \quad \text{to nearest centimeter}$$

(ii) Area =

$$9\sqrt{3} + 3(6\pi - 9\sqrt{3})$$
$$18\pi - 18\sqrt{3}$$
$$18(\pi - \sqrt{3})$$
$$= 25cm^2 \text{ to the nearest whole number}$$

Question 7

(a) (i) reflection in y=x

(ii) sketch showing mirror image about y=x

(b) (i) stretch parallel to the y-axis, by a factor of 3

(ii) Let

$$y = 3\ln x$$
$$e^y = x^3$$
$$x = \sqrt[3]{e^y}$$

so

$$f^{-1}(x) = e^{\frac{1}{3}x}$$

(Note : The last line has swopped around the x and y from the previous line. This is acceptable because the expression on the penultimate line is just expressing the relationship between two variables - we can choose what these two variables are at will. The reason we swop the x and y around is that we can then draw the graph of the inverse function on the same axes as the original function. If we kept it in its original form of $x = \sqrt[3]{e^y}$, you would need to graph it with x denoting the vertical axis and y the horizontal)

Pure Maths 2 - MAP2, 14 June 2002

Question 1

$x+1$ $\quad$ x^2+x-6
x^3+2x^2-5x-6
x^3+x^2
− − − − − − − − − − −
x^2-5x-6
x^2+x
− − − − − − − − − − − − − − − −
$-6x-6$
$-6x-6$
− −

(Sorry about the alignment for this question)

Question 2

(a) (i) Using given data, can form

$$14.6 = a + 13b \quad (1)$$

$$13 = a + 5b \quad (2)$$

Subtract (2) from (1)

$$1.6 = 8b$$

$$b = \frac{1.6}{8} = 0.2$$

therefore

$$a = 14.6 - 13(0.2)$$

$$a = 12$$

(ii)

$$p_3 = a + bp_2$$

$$p_3 = 12 + (0.2)(14.6)$$

$$= 14.92$$

(b) Set $p_{t+1} = p_t$

$$p_t = 12 + 0.2p_t$$

$$0.8p_t = 12$$

$$p_t = \frac{12}{0.8} = 15$$

Question 3

Sketch
obvious discontinuity at x=2

$$y = \frac{x}{x-2} = \frac{1}{1-\frac{2}{x}}$$

As $x \to \infty$

$$y \to 1$$

$y = 1$ and $x = 2$ are the asymptotes
It goes thru (0,0)

Question 4

Volume

$$= \pi \int_1^2 \left(x - \frac{1}{x}\right)^2 dx$$

$$= \pi \int_1^2 \left(x^2 - 2 + \frac{1}{x^2}\right) dx$$

$$= \pi \left[\frac{x^3}{3} - 2x - \frac{1}{x}\right]_1^2$$

$$= \pi \left\{\left(\frac{8}{3} - 4 - \frac{1}{2}\right) - \left(\frac{1}{3} - 2 - 1\right)\right\}$$

$$= \pi \left(-\frac{11}{6} - \left(-\frac{8}{3}\right)\right)$$

$$= \frac{5}{6}\pi$$

Question 5

(a) (i)

$$\frac{dy}{dx} = x\sec^2 x + \tan 3x$$

(ii)

$$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$$

(b)

$$\int_0^{\frac{\pi}{8}} x \sin 2x \, dx$$

$$= \left[x\left(-\frac{\cos 2x}{2}\right)\right]_0^{\frac{\pi}{8}} + \int_0^{\frac{\pi}{8}} \left(\frac{\cos 2x}{2}\right)$$

$$= \frac{\pi}{8}.\left(-\frac{1}{2\sqrt{2}}\right) + \left[\frac{\sin 2x}{4}\right]_0^{\frac{\pi}{8}}$$

$$= \frac{\pi}{8}.\left(-\frac{1}{2\sqrt{2}}\right) + \frac{1}{4\sqrt{2}}$$

$$= \frac{4-\pi}{16\sqrt{2}}$$

Question 6

(a) Radius

$$= \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

Coordinates of center

$$(-1, 3)$$

(b) Differentiate equation

$$2x + 2y\frac{dy}{dx} + 2 - 6\frac{dy}{dx} = 0$$

$$(2y-6)\frac{dy}{dx} = -2 - 2x$$

$$\frac{dy}{dx} = \frac{-2-2x}{2y-6}$$

At x=2, y=4

$$\frac{dy}{dx} = \frac{-2-4}{8-6} = -3$$

Equation of tangent

$$y - y_0 = -3(x - x_0)$$

$$y - 4 = -3(x - 2)$$

$$y = -3x + 10$$

Question 7

(a)

$$PQ = 2r\cos\theta$$

Height of triangle OPQ

$$r\sin\theta$$

Therefore area of OPQ

$$= r\cos\theta . r\sin\theta$$

$$= r^2\cos\theta\sin\theta$$

$$= \frac{r^2\sin 2\theta}{2}$$

(*Using* $\sin 2\theta = 2\sin\theta\cos\theta$)

(b) Area of $\triangle OPQ$ + area of sectors OAP and OBQ = $\frac{1}{2}\times$ the area of a semi-circle

$$\frac{r^2\sin 2\theta}{2} + 2\left(\frac{r^2\theta}{2}\right) = \frac{1}{2}\frac{\pi r^2}{2}$$

$$2r^2\sin 2\theta + 4r^2\theta = \pi r^2$$

$$2\sin 2\theta + 4\theta = \pi$$

(c)

$$f = 4\theta + 2\sin 2\theta - \pi$$

$$f' = 4 + 4\cos\theta$$

second approximation

$$x_1 = 0.5 - \frac{4(0.5) + 2\sin 1}{4 + 4\cos 0.5}$$

$$x_1 = 0.5 - \frac{0.54134...}{6.16120...}$$

$$x_1 = 0.412 \text{ to 3 d.p.}$$

Question 8

(a)

$$\frac{\cot^2\theta}{1+\cot^2\theta}$$

$$= \frac{\frac{\cos^2\theta}{\sin^2\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}}$$

$$= \frac{\cos^2\theta}{\sin^2\theta + \cos^2\theta}$$

$$= \cos^2\theta$$

(b) equation becomes

$$\cos^2\theta = 2\sin 2\theta$$

$$\cos^2\theta = 2\times 2\sin\theta\cos\theta$$

$$\cos\theta(cos\theta - 4\sin\theta) = 0$$

$$\cos\theta = 0 \Rightarrow \theta = 90°, 270°$$

$$cos\theta - 4\sin\theta = 0 \Rightarrow \tan\theta = \frac{1}{4} \Rightarrow \theta = 14°, 194°$$

Pure Maths 3 - MAP3, 20 June 2002

Question 1

(a)

$$x = 1 - t^2 \Rightarrow \frac{dx}{dt} = -2t$$

$$y = 2t \Rightarrow \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= 2.\left(-\frac{1}{2t}\right)$$

$$= -\frac{1}{t}$$

(b) At t=3

$$\frac{dy}{dx} = -\frac{1}{3}$$

so gradient of normal at this point

$$= 3$$

so equation of normal at t=3

$$y - 6 = 3(x - (-8))$$

$$y = 3x + 24 + 6$$

$$y = 3x + 30$$

Question 2

(a)

$$\frac{4-x}{(1-x)(2+x)} = \frac{A}{1-x} + \frac{B}{2+x}$$

$$4 - x = A(2+x) + B(1-x)$$

Compare coefficients of x

$$-1 = A - B$$

Let x = 1

$$3 = 3A \Rightarrow A = 1$$

and

$$B = A + 1 = 2$$

so required form is

$$\frac{1}{1-x} + \frac{2}{2+x}$$

(b) (i)

$$(2+x)^{-1} = 2^{-1}\left(1 + \frac{x}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1 - \frac{x}{2} + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2\right)$$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$$

(ii)

$$(1-x)^{-1} = 1 + x + \frac{(-1)(-2)}{2!}(-x)^2$$

$$= 1 + x + x^2$$

(c)

$$(4-x)\left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}\right)(1 + x + x^2)$$

$$= 2 + \frac{x}{2} + \frac{5x^2}{4}$$

Question 3

(a)

$$\frac{4}{9} + \frac{y^2}{25} = 1$$

$$y^2 = 25 - \frac{100}{9}$$

$$y = \pm 3.73$$

(b) differentiating equation

$$\frac{2x}{9} + \frac{2y}{25}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{9}.\frac{25}{2y}$$

$$\frac{dy}{dx} = -\frac{25x}{9y}$$

At $x = 2, y = \pm 3.73$

$$\frac{dy}{dx} = -\frac{75}{\pm 33.6}$$

$$= \pm 1.49$$

Question 4

(a)

$$\frac{1}{2}m_0 = m_0 e^{-28k}$$

$$e^{-28k} = \frac{1}{2}$$

$$-28k = \ln\frac{1}{2}$$

$$k = 0.024755(2)....$$

(b)

$$1 = m_0 e^{-0.024755(100)}$$

$$m_0 = e^{2.4755}$$

$$m_0 = 11.9g$$

Question 5

(a)

$$\frac{dy}{dx} = \frac{1}{y^2}$$
$$\int y^2 dy = \int dx$$
$$\frac{y^3}{3} = x + c$$
$$y = \sqrt[3]{3x + D}$$

(b) When x=1, y = -1

$$-1 = \sqrt[3]{3 + D}$$
$$-1 = 3 + D$$
$$D = -4$$

so particular solution is

$$y = \sqrt[3]{3x - 4}$$

Question 6

(a) (i)

$$f(x) = \ln(1 - 2x)$$
$$f'(x) = -2\frac{1}{(1 - 2x)}$$
$$f''(x) = -4\frac{1}{(1 - 2x)^2}$$
$$f'''(x) = -16\frac{1}{(1 - 2x)^3}$$

(ii)

$$f(0) = \ln(1) = 0$$
$$f'(0) = -2.\frac{1}{(1)} = -2$$
$$f''(0) = -4.\frac{1}{(1)^2} = -4$$
$$f'''(0) = -16.\frac{1}{(1)^3} = -16$$

so

$$\ln(1 - 2x) = 0 - 2x - 4\frac{x^2}{2!} - 16\frac{x^3}{3!}$$
$$\ln(1 - 2x) = -2x - 2x^2 - \frac{8}{3}x^3$$

(b)

$$2xe^x = 2x\left(1 + x + \frac{x^2}{2} + ...\right)$$
$$\approx 2x + 2x^2 + x^3$$

so

$$2xe^x + \ln(1 - 2x) = 2x + 2x^2 + x^3 - 2x - 2x^2 - \frac{8}{3}x^3$$
$$= -\frac{5}{3}x^3$$

so

$$k = -\frac{5}{3}$$

Question 7

(a) (i) Two non-equal sides must add up to $\frac{20}{2} = 10$m. So if one side is x, the other will be $10 - x$. So

$$A = x(10 - x)$$

(ii)

$$\frac{dA}{dx} = (10 - x) + x(-1)$$
$$= 10 - 2x$$

(b) from above

$$\frac{dA}{dt}\frac{dt}{dx} = 10 - 2x$$
$$\frac{dA}{dt}\frac{1}{0.5} = 10 - 2(4)$$
$$\frac{dA}{dt} = 1\ m^2 s^{-1}$$

Question 8

(a)

$$3 + 4t = 8 - s \quad (3)$$
$$-2 + 4t = -1 + 3s \quad (4)$$
$$1 + 3t = 2 + 2s \quad (5)$$

(4) from (3)

$$5 = 9 - 4s$$
$$4s = 4$$
$$s = 1$$

put this into (3)

$$3 + 4t = 8 - 1$$
$$4t = 4$$
$$t = 1$$

so coordinates of intersection are

$$(7, 2, 4)$$

This is consistent with both line equations

(b) (i)

$$\begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} . \begin{bmatrix} 1 \\ 11 \\ -16 \end{bmatrix} = 0$$
$$\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} . \begin{bmatrix} 1 \\ 11 \\ -16 \end{bmatrix} = 0$$

(ii) The equation of a plane is

$$\mathbf{r}.\mathbf{n} = \mathrm{d}$$

where $\mathbf{n}$ is given from (b)(i) (and $\mathbf{r} = [\mathrm{x, y, z}]^{\mathrm{T}}$)

so equation becomes

$$x + 11y - 16z = d$$

The point $(3, -2, 1)$ lies on the plane, so

$$3 - 22 - 16 = d \Rightarrow d = -35$$

Cartesian Equation is therefore

$$x + 11y - 16z = -35$$

Pure Maths 4 - MAP4, 14 June 2002

Question 1

$$-2-4i$$

is also a root, being the complex conjugate of $-2+4i$.

$$(-2-4i)(-2+4i)\alpha = 80$$

$$(4+16)\alpha = 80$$

$$\alpha = \frac{80}{20} = 4$$

(Here I have used : product of roots = $(-1)^n C$, where C is the last (i.e. the constant term). I could also have used : sum of terms = 0 (i.e. the coefficient of x^2))

putting x = 4 into the cubic

$$4^3 + 4k - 80 = 0$$

$$4k = 80 - 64$$

$$k = \frac{16}{4} = 4$$

Question 2

$$i(a+ib)+4 = (2-i)(a-ib)$$

$$ia - b + 4 = 2a - 2ib - ia - b$$

$$-b + 4 - 2a + b = -2ib - ia - ia$$

$$4 - 2a = -i(2a+2b)$$

Comparing real parts

$$4 - 2a = 0$$

$$a = 2$$

Comparing imaginary parts

$$2a + 2b = 0$$

$$4 + 2b = 0$$

$$b = -2$$

Therefore

$$z = 2 - 2i$$

Question 3

(a)

$$y = 2\sqrt{x}$$

$$\dot{y} = \frac{1}{2}.2x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$$

therefore

$$\dot{y}^2 = x^{-1}$$

Now

$$\frac{ds}{dx} = \sqrt{1 = \dot{y}^2}$$

$$= \sqrt{1 + x^{-1}}$$

The surface area generated

$$S = 2\pi \int_0^3 2\sqrt{x}\frac{ds}{dx}dx$$

$$= 4\pi \int_0^3 \sqrt{x}\sqrt{1+x^{-1}}dx$$

$$= 4\pi \int_0^3 \sqrt{1+x}dx$$

(b) integrating the above

$$= 4\pi.\frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^3$$

$$= \frac{8\pi}{3}\{(8)-(1)\}$$

$$= \frac{56\pi}{3}$$

Question 4

(a)

$$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

so

$$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

(b)

$$r = 2, \quad \theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$\sqrt{3} + i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$\sqrt{3} - i = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

(c)

$$(\sqrt{3}+i)^n - (\sqrt{3}-i)^n = 0$$

$$2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right) - 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right) = 0$$

$$2^n\left(2i\sin\frac{n\pi}{6}\right) = 0$$

$$\sin\frac{n\pi}{6} = 0$$

$$\frac{n\pi}{6} = 0, \pi, 2\pi, 3\pi, ...$$

$$n\pi = 0, 6\pi, 12\pi, 18\pi, ...$$

$$n = 0, 6, 12, 18, ...$$

i.e.

$$n = 6k; \quad k = 0, 1, 2, ..$$

Question 5

(a) Argand Diagram

(b) (i) diameter (d) is given by

$$d^2 = 4^2 + 2^2 = 20 \Rightarrow d = \sqrt{20}$$

radius

$$= \frac{\sqrt{20}}{2} = \sqrt{5}$$

center is at

$$\frac{5+1}{2} + i\frac{-1-3}{2} = 3 - 2i$$

(ii) equation of circle is

$$|z - z_0| = k$$

here

$$|z - (3 - 2i)| = \sqrt{5}$$

$$|z - 3 + 2i| = \sqrt{5}$$

Question 6

<u>Assume true for n=k</u> - then if we add another term

$$\sum_{r=1}^{k+1} \frac{1}{(3r-2)(3r+1)} = \frac{k}{3k+1} + \frac{1}{(3(k+1)-2)(3(k+1)+1)}$$

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)}{(3k+1)(3k+4)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$= \frac{k+1}{3k+4}$$

$$= \frac{k+1}{3(k+1)+1}$$

<u>Check for n=1</u>

$$\frac{1}{(3-2)(3+1)} = \frac{1}{3+1}$$

$$\frac{1}{4} = \frac{1}{4}$$

Question 7

(a)

$$y = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$$

$$\frac{dy}{dx} = \frac{1}{2}.\frac{1}{(1+x)} + \frac{1}{2}.\frac{1}{(1-x)}$$

$$= \frac{1}{2}.\frac{(1-x)+(1+x)}{(1+x)(1-x)}$$

$$= \frac{1}{1-x^2}$$

(b) (i)

$$\int \tanh^{-1} x \, dx$$

$$= \int 1.\tanh^{-1} \, dx$$

$$= x\tanh^{-1} x - \int \frac{x}{1-x^2}\, dx$$

$$= x\tanh^{-1} x + \frac{1}{2}\int \frac{-2x}{1-x^2}\, dx$$

$$= x\tanh^{-1} + \frac{1}{2}\ln(1-x^2) + c$$

(ii)

$$\int_0^{\frac{1}{3}} \tanh^{-1} x = \left[x\tanh^{-1} x + \frac{1}{2}\ln\left(1-x^2\right)\right]_0^{\frac{1}{3}}$$

$$= \left(\frac{1}{3}\tanh^{-1}\frac{1}{3} + \frac{1}{2}\ln\left(1-\frac{1}{9}\right)\right) - (0)$$

$$= \frac{1}{3}.\frac{1}{2}\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) + \frac{1}{2}\ln\left(\frac{8}{9}\right)$$

$$= \frac{1}{6}\ln(2) + \frac{1}{2}\ln 8 - \frac{1}{2}\ln 9$$

$$= \frac{1}{6}\ln 2 + \frac{3}{2}\ln 2 - \ln 3$$

$$= \frac{5}{3}\ln 2 - \ln 3$$

Pure Maths 5 - MAP5, 30 May 2002

Question 1

$$Area = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} \left(a \sin \frac{1}{2}\theta\right)^2 d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi} \left(\sin^2 \frac{1}{2}\theta\right) d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi} \frac{1}{2}(1-\cos\theta) d\theta$$
$$= \frac{a^2}{4}\left[\theta - \sin\theta\right]_0^{2\pi}$$
$$= \frac{a^2}{4}\{(2\pi)-(0)\} = \frac{a^2\pi}{2}$$

Question 2

$$\frac{dy}{dx} + \frac{2}{x}y = x^2$$
$$IF = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

equation becomes

$$x^2\frac{dy}{dx} + 2xy = x^4$$
$$\frac{d}{dx}\left(x^2 y\right) = x^4$$
$$x^2 y = \frac{x^5}{5} + C$$
$$y = \frac{x^3}{5} + \frac{C}{x^2}$$

Question 3

(a)

$$\cos\theta = \frac{x}{r}$$
$$\sin\theta = \frac{y}{r}$$

(b) (i)

$$r = 2\cos\theta - 4\sin\theta$$
$$r = 2\frac{x}{r} - 4\frac{y}{r}$$
$$r^2 = 2x - 4y$$

since $r^2 = x^2 + y^2$

$$x^2 + y^2 = 2x - 4y$$

(ii) equation can be converted to

$$(x-1)^2 + (y+2)^2 = 5$$

this has center (1,-2) and radius $\sqrt{5}$

Question 4

(a)

$$\int \frac{\sin\theta}{\sqrt{\cos\theta}} d\theta$$

Let $x = \cos\theta$, $\frac{dx}{d\theta} = -\sin\theta$

$$\int -\frac{dx}{d\theta}\frac{1}{\sqrt{x}} d\theta$$
$$= -\int \frac{1}{\sqrt{x}} dx$$
$$= -2\sqrt{x} + c$$
$$= -2\sqrt{\cos x} + c$$

(b) (i) Denominator becomes zero when $\theta = \frac{\pi}{2}$

(ii)

$$\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{\cos\theta}} d\theta = \left[-2\sqrt{\cos\theta}\right]_0^{\frac{\pi}{2}}$$
$$= (0) - (-2) = 2$$

Question 5

(a) Auxiliary Equation

$$m^2 + 4 = 0$$
$$m^2 = -4 \Rightarrow m = \pm 2i$$

Complementary Function

$$y = A\cos 2x + B\sin 2x$$

Particular Integral

Try

$$y = C\sin x + D\ cosx$$
$$y' = C\cos x - D\sin x$$
$$y'' = -C\sin\ x - D\cos x$$

Insert into D.E.

$$-C\sin x - D\cos x + 4C\sin x + 4D\cos x = 6\cos x$$

Comparing coefficients of $\sin x$

$$-C + 4C = 0 \Rightarrow C = 0$$

Comparing coefficients of $\cos x$

$$-D + 4D = 6$$
$$3D = 6 \Rightarrow D = 2$$

PI is then

$$y = 2\cos x$$

General Solution

$$y = A\cos 2x + B\sin 2x + 2\cos x$$

(b) (i) Let

$$y = \lambda x \sin 2x$$
$$y' = \lambda x.2\cos 2x + \lambda \sin 2x$$
$$= \lambda(2x.\cos 2x + \sin 2x)$$
$$y'' = \lambda(2x(-2\sin 2x) + 2\cos 2x + 2\cos 2x)$$

Insert into D.E.

$$-4\lambda x \sin x + 2\lambda \cos 2x + 2\lambda \cos 2x + 4\lambda x \sin 2x = 6\cos 2x$$

Comparing coefficients of $\cos 2x$

$$2\lambda + 2\lambda = 6$$
$$4\lambda = 6$$
$$\lambda = \frac{3}{2}$$

(ii) General Solution

$$y = A\cos 2x + B\sin 2x + \frac{3}{2}x\sin 2x$$

Question 6

(a)

$$y_{1.1} \approx y_{0.9} + 2(0.1)(0.5)$$
$$y_{1.1} - y_{0.9} \approx 0.1 \qquad (6)$$

(b) (i)

$$y_{1.1} \approx 2y_1 - y_{0.9} + (0.1)^2 e^{\cos 1 \cos 1}$$
$$y_{1.1} + y_{0.9} \approx 2 + (0.1)^2 e^{\cos 1 \cos 1}$$
$$y_{1.1} + y_{0.9} \approx 2.01339(0).... \qquad (7)$$

(ii) Adding (6) and (7)

$$2y_{1.1} \approx 2.11339(0)...$$
$$y_{1.1} \approx \frac{1}{2}(2.11339(0)...) \approx 1.057 \text{ (to 3 d.p.)}$$

(c)

$$y_{1.2} \approx 2y_{1.1} - y_1 + h^2 y''_{1.1}$$
$$y_{1.2} \approx 2(1.057) - 1 + (0.1)^2 e^{\cos 1.1 \cos 1.057}$$
$$\approx 1.126 \text{ (to 3 d.p.)}$$

Question 7

(a)

$$\cos x - 1 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) - 1$$
$$= -\frac{x^2}{2} + \frac{x^4}{24}$$

(b)

$$e^{\cos x - 1} = e^{-\frac{x^2}{2} + \frac{x^4}{24}}$$
$$= 1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \frac{\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2}{2}$$
$$1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} + ...$$
$$1 - \frac{x^2}{2} + \frac{x^4}{6}$$

(c) As $\sin^2 x \approx x^2$ for very small angles

$$\lim_{x\to 0} \frac{1 - e^{(\cos x - 1)}}{\sin^2 x}$$
$$= \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^4}{6}}{x^2}$$
$$= \lim_{x\to 0} \left(\frac{1}{2} - \frac{x^2}{6}\right)$$
$$= \frac{1}{2}$$

Pure Maths 6 - MAP6, 30 May 2002

Question 1

(a)

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \mathbf{i}(-1-8) - \mathbf{j}(-3-4) + \mathbf{k}(6-1)$$
$$= -9\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

$$(\mathbf{a}.\mathbf{b}).\mathbf{c} = \begin{bmatrix} -9 \\ 7 \\ 5 \end{bmatrix} . \begin{bmatrix} 8 \\ 1 \\ 13 \end{bmatrix} = -72 + 7 + 65 = 0$$

(b) coplanar (or linearly dependent)

Question 2

(a) $\mathbf{M}^2$ can only be calculated for a square matrix

(b) (i)

$$\mathbf{M}(\mathbf{M} - 6\mathbf{I}) = \mathbf{M}^2 - 6\mathbf{I}\mathbf{M} = \mathbf{M}^2 - 6\mathbf{M} = 5\mathbf{I}$$

(ii)

$$\mathbf{M}(\mathbf{M} - 6\mathbf{I}) = 5\mathbf{I}$$
$$\mathbf{M}^{-1}\mathbf{M}(\mathbf{M} - 6\mathbf{I}) = \mathbf{M}^{-1}5\mathbf{I}$$
$$(\mathbf{M} - 6\mathbf{I}) = 5\mathbf{M}^{-1}$$
$$\mathbf{M}^{-1} = \frac{1}{5}\mathbf{M} - \frac{6}{5}\mathbf{I}$$

Question 3

(a) **A** - shear parallel to the y-axis. (Produce example using an arbitrary point to show displacement of 3 units in y-direction)

B rotation of 45° about the x-axis

(b) **A** y-axis **B** x-axis

Question 4

(a) $\mathbf{c} \times \mathbf{r} = 0$ implies that $\mathbf{c}$ is a line parallel to $\mathbf{r}$. The locus of P is therefore a straight line through the origin, parallel to $\mathbf{c}$.

(b)

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$$

where $\mathbf{b}$ can be derived from the denominators. i.e.

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

and $\mathbf{a}$ could be derived from the numbers in the numerator, i.e.

$$\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$$

Question 5

(a)

$$\mathbf{b} - \mathbf{a} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{c} - \mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{vmatrix}$$
$$= \mathbf{i}(3-2) - \mathbf{j}(-2+1) + \mathbf{k}(4-3)$$
$$= \mathbf{i} + \mathbf{j} + \mathbf{k}$$

(b) Area of triangle

$$= \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \text{ X } (\mathbf{c} - \mathbf{a})|$$
$$= \frac{1}{2}\sqrt{1^2 + 1^2 + 1^2} = \frac{1}{2}\sqrt{3}$$

Question 6

(a)

$$|\mathbf{A}| = 3(-\text{a} - 10) - 1(2\text{a} - 5) + 8(4 - (-1))$$
$$= -3a - 30 - 2a + 5 + 40$$
$$= 15 - 5a$$

(b) $\mathbf{M}$ is singular when

$$15 - 5a = 0$$
$$5a = 15$$
$$a = 3$$

(c) (i) when $a = 2$

$$\mathbf{M} = \begin{bmatrix} 3 & 1 & 8 \\ 2 & -1 & 5 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -12 & 1 & 5 \\ 14 & -2 & -5 \\ 13 & 1 & -5 \end{bmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} -12 & 14 & 13 \\ 1 & -2 & 1 \\ 5 & -5 & -5 \end{bmatrix}$$

and $|\mathbf{M}| = 15 - 5(2) = 5$

$$\mathbf{M}^{-1} = \frac{1}{5}\begin{bmatrix} -12 & 14 & 13 \\ 1 & -2 & 1 \\ 5 & -5 & -5 \end{bmatrix}$$

(ii)

$$\mathbf{M}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} = \mathbf{M}^{-1}\mathbf{b}$$
$$\mathbf{x} = \frac{1}{5}\begin{bmatrix} -12 & 14 & 13 \\ 1 & -2 & 1 \\ 5 & -5 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Question 7

(a) (i)

$$\begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix}$$

$$(3-\lambda)(-2-\lambda)+4=0$$
$$-6-3\lambda+2\lambda+\lambda^2+4=0$$
$$\lambda^2-\lambda-2=0$$
$$(\lambda+1)(\lambda-2)=0$$
$$\lambda=-1,2$$

(ii) For $\lambda=-1$

$$\begin{aligned} 4x-y &= 0 \\ 4x-y &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

For $\lambda=2$

$$\begin{aligned} x-y &= 0 \\ 4x-4y &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b)

$$\mathbf{U}=\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \quad \mathbf{D}=\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Mechanics 1 - MAM1, 23 May 2002

Question 1

(a)

$$v^2 = u^2 + 2gs$$

$$v^2 = 0 + 2(9.8)(22.5)$$

$$v = \sqrt{441} = 21ms^{-1}$$

(b)

$$t = \frac{v-u}{a} = \frac{21}{9.8} = 2.14s$$

(c) Appropriate straight line, starting from origin

(d) Ignored air resistance, treated as particle

Question 2

(a) Conservation of Momentum

$$0.3\begin{bmatrix}7\\4\end{bmatrix} = 0.3\begin{bmatrix}v_1\\v_2\end{bmatrix} + 0.5\begin{bmatrix}6\\0\end{bmatrix}$$

$$\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}7\\4\end{bmatrix} - \frac{0.5}{0.3}\begin{bmatrix}6\\0\end{bmatrix} = \begin{bmatrix}-3\\4\end{bmatrix}$$

(b) Speed of A (v) is given by

$$v^2 = (-3)^2 + (4)^2 \Rightarrow v = 5ms^{-1}$$

(c) Speed of B = $6ms^{-1}$ so this is the fastest

Question 3

(a) Distance traveled during period of constant speed = 60m

Distance during deceleration is derived from

$$v^2 = u^2 + 2as$$

$$0 = 3^2 + 2(-0.2)s$$

$$s = \frac{9}{0.4} = 22.5$$

Total Distance = 60 + 22.5 = 82.5m

(b) time to decelerate to 0 ms^{-1}

$$t = \frac{v-u}{a} = \frac{-3}{-0.2} = 15s$$

Total time = 20 + 15 = 35s

(c) Average Speed

$$= \frac{\text{Distance}}{\text{Time}} = \frac{82.5}{35} = 2.36ms^{-1}$$

Question 4

(a)

$$s = ut + \frac{1}{2}at^2$$

$$0 = (20\sin 53)t + \frac{1}{2}(-9.8)t^2$$

$$t(20\sin 53 - 4.9t) = 0$$

giving t=0 or

$$4.9t = 20\sin 53$$

$$t = \frac{20\sin 53}{4.9}$$

$$= 3.26s$$

(Alternatively, you could the symmetry of the situation and use the method below, although it is not as mathematically elegant)

Vertical velocity

$$= 20\sin 53$$

Time to reach max height is given by

$$t = \frac{v-u}{a} = \frac{-20\sin 53}{-9.8} = 1.63s$$

By symmetry, required time to reach X

$$= 1.63 \times 2 = 3.26s$$

(b) Distance OX

$$= 20\cos 53 \times 3.26$$

$$= 39.2m$$

(c) (i) Least Speed

$$= 20\cos 53 = 12.0ms^{-1} \text{ to 3 sig. figs}$$

(ii) Horizontal

Question 5

(a) Reaction R at A = 0.5g.

Friction

$$= \frac{2}{7}R = \frac{2}{7}(0.5g) = 1.4N$$

(b) Resolving vertically

$$0.2g - T = 0.2a \qquad (8)$$

Resolving horizontally

$$T - 1.4 = 0.5a \qquad (9)$$

Adding (8) and (9)

$$0.2g - 1.4 = 0.7a$$

$$a = \frac{0.2g - 1.4}{0.7} = 0.8ms^{-1}$$

(c) From (8)

$$1.96 - T = 0.2(0.8)$$
$$T = 1.96 - 0.16$$
$$= 1.8N$$

(d)

$$s = ut + \frac{1}{2}at^2$$
$$0.625 = \frac{1}{2}(0.8)t^2$$
$$t^2 = \frac{0.625}{0.4} = 0.15625$$
$$t = 1.25s$$

Question 6

(a)

$$\frac{x}{\sin 10} = \frac{50}{\sin 50}$$
$$x = \frac{50\sin 10}{\sin 50} = 11.3kmh^{-1}$$

(b) (i) Northerly component

$$= 50\cos 10 = 49.2kmh^{-1}$$

(ii)

$$\frac{R}{\sin 120} = \frac{50}{\sin 50}$$
$$R = \frac{50\sin 120}{\sin 50} = 56.5kmh^{-1}$$

The question is implying that you add the result from b(i) to $11.3\cos 50$, but I would suggest the method above is better

Question 7

(a)

$$\mathbf{r} = \begin{bmatrix} 2t^2 + 6 \\ 5t \end{bmatrix}$$
$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}} = \begin{bmatrix} 4t \\ 5 \end{bmatrix}$$

(b)

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

From $F = ma$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = m\begin{bmatrix} 4 \\ 0 \end{bmatrix} \Rightarrow m = \frac{1}{2}$$

(c) (i) Additional force implies addditional acceleration, thus

$$\mathbf{F} = \begin{bmatrix} 0 \\ t \end{bmatrix} \Rightarrow \mathbf{a}\begin{bmatrix} 2 \\ 2t \end{bmatrix}$$

Resultant acceleration (at t=5)

$$\mathbf{a} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2t \end{bmatrix} = \begin{bmatrix} 4 \\ 2t \end{bmatrix}$$

(ii) Integrating

$$\mathbf{a} = \begin{bmatrix} 4 \\ 2t \end{bmatrix}$$

gives

$$\mathbf{v} = \begin{bmatrix} 4t \\ t^2 \end{bmatrix} + \mathbf{c}$$

When t=5,

$$\mathbf{v} = \begin{bmatrix} 20 \\ 25 \end{bmatrix} + \mathbf{c}$$

From (a), at t=5

$$\mathbf{v} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$$
$$\Rightarrow \mathbf{c} = \begin{bmatrix} 0 \\ -20 \end{bmatrix}$$

so

$$\mathbf{v} = \begin{bmatrix} 4t \\ t^2 - 20 \end{bmatrix}$$

When t=10,

$$\mathbf{v} = \begin{bmatrix} 40 \\ 80 \end{bmatrix}$$

Mechanics 2 - MAM2, 18 June 2002

Question 1

(a) Diagram showing R and S as the two upward forces at C and D respectively

(b) Resolving vertically

$$R + S = 60g + 10g = 70g$$

Taking moments about C

$$60g(0.5) + 10g(1) = S(2)$$

$$30g + 10g = 2S$$
$$S = 20g$$
$$R = 50g$$

Question 2

(a)

$$\omega = 2\pi f = 2\pi \frac{8}{60} = \frac{4\pi}{15}$$

(b) From $v = r\omega$

$$r = \frac{v}{\omega} = \frac{0.75 \times 15}{4\pi} = 0.90m \text{ to 2 sig figs}$$

(c)

$$a = r\omega = v\omega = 0.75 \times \frac{4\pi}{15} = \frac{\pi}{5}$$

(This method avoids any rounding error)

Question 3

(a) Diagram showing

- force of 1960N downwards parallel to the slope
- the weight W of 1200g
- the reaction N perpendicular to the slope
- and the tractive force F_T up the slope and parallel to it

(b) (i) When velocity is constant

$$F_T = 1960 + 1200g \sin 5^\circ$$

$$= 2984.95 \approx 2985N$$

(ii)

$$P = Fv$$

$$= 2985 \times 15$$

$$= 44775W$$

(c) If the tractive force remains the same as in (b)(i)

$$v_{max} = \frac{P_{max}}{F} = \frac{60 \times 10^3}{2985} = 20.1ms^{-1}$$

Question 4

Diagram with two masses A and B both of mass m. Take rightwards as positive.

For A,

- initial velocity = U to the right
- final velocity V_A to the right

For B

- initial velocity $3U$ to the left
- final velocity V_B to the right

(a) (i) Conservation of Momentum

$$U - 3U = V_A + V_B$$

$$V_A + V_B = -2U \qquad (10)$$

Coefficient of Restitution

$$e(3U + U) = V_B - V_A$$

$$V_B - V_A = 4Ue \qquad (11)$$

Adding (10) and (11)

$$2V_B = -2U + 4Ue$$

$$V_B = U(2e - 1)$$

and from(10)

$$V_A = -2U - U(2e - 1)$$

$$= -U(2 + 2e - 1)$$
$$= -U(1 + 2e)$$

i.e. V_A has magnitude $U(1 + 2e)$ (to the left)

(ii)

$$-U(1 + 2e)U(2e - 1) > 0$$

$$-U^2(2e - 1 + 4e^2 - 2e) > 0$$
$$-U^2(4e^2 - 1) > 0$$
$$4e^2 - 1 < 0$$
$$4e^2 < 1$$
$$e^2 < \frac{1}{4}$$
$$|e| < \frac{1}{2}$$

(b) (i) Impulse = Momentum of B after - Momentum of B before

$$I = mU(2e - 1) - m(-3U)$$
$$= 2mUe - mU + 3mU$$
$$= 2mUe + 2mU$$
$$= 2mU(e + 1)$$

(ii) When perfectly elastic (e=1)

Impulse = 4mU

Question 5

(a) When $t = \frac{\pi}{4}$

$$\mathbf{F} = \begin{bmatrix} 4\sin\frac{\pi}{2} \\ -6\cos\pi \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

(b) From (a)

$$\mathbf{a} = \begin{bmatrix} 2\sin 2 \\ -3\cos 4t \end{bmatrix}$$

so

$$\mathbf{v} = \begin{bmatrix} -\cos 2t \\ -\frac{3}{4}\sin 4t \end{bmatrix} + \mathbf{C}$$

When $t = 0, \mathbf{v} = [0 \;\; 3]^{\mathrm{T}}$

$$\mathbf{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mathbf{C} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\Rightarrow \mathbf{C} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \mathbf{v} = \begin{bmatrix} 1 - \cos 2t \\ 3 - 0.75\sin 4t \end{bmatrix}$$

(c) When $t = \frac{\pi}{4}$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

now

$$P = \mathbf{F}.\mathbf{v}$$
$$= \begin{bmatrix} 4 \\ 6 \end{bmatrix} . \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= 4 + 18 = 22W$$

Question 6

(a)

$$mgh + \frac{1}{2}mv^2 = \frac{1}{2}mu^2$$

$$gr(1 - \cos\theta) + \frac{1}{2}v^2 = \frac{1}{2}\left(\sqrt{\frac{7gr}{2}}\right)^2$$

where θ is the angle between B and the skater

$$\frac{1}{2}v^2 = \frac{1}{2}\left(\frac{7gr}{2}\right) - gr\left(\frac{3}{2}\right)$$

$$v^2 = \frac{7gr}{2} - 3gr = \frac{gr}{2}$$

$$v = \sqrt{\frac{gr}{2}}$$

(b)

$$N + mg\cos 60 = \frac{mv^2}{r}$$

$$N = \frac{mgr}{r.2} - \frac{mg}{2}$$
$$N = 0$$

since reaction = zero, this implies contact is about to be lost

Mechanics 3 - MAM3, 24 June 2002

Question 1

(a) Diagram showing

- Weight 12g
- horizontal force T, the tension in the bar
- the reaction R in arbitrary direction from A

(b) Taking moments about A

$$W \sin 30(0.75) = T \cos 30(0.5)$$

$$12g\left(\frac{1}{2}\right)(0.75) = T\frac{\sqrt{3}}{2}(0.5)$$

$$\frac{9}{2}g = \frac{\sqrt{3}}{4}T$$

$$T = \frac{18g}{\sqrt{3}} = 6\sqrt{3}g\ N$$

Question 2

(a) (i) A constant couple implies a constant acceleration (or deceleration). Since the gradient of $\frac{d\theta}{dt}$ represents the acceleration, this line will be a straight line.

(ii)

$$\ddot{\theta} = \frac{\dot{\theta}_f - \dot{\theta}_i}{t} = \frac{0 - 20\pi}{100} = -\frac{\pi}{5} rad\ s^{-2}$$

(iii)

$$\theta_{tot} = \frac{(\dot{\theta}_f + \dot{\theta}_i)}{2} \times t$$

$$= \frac{0 + 20\pi}{2} \times 100$$

$$= 1000\pi$$

(b) (i)

$$T = I\ddot{\theta}$$

where T is the torque/couple

$$-10\pi = I\left(-\frac{\pi}{5}\right)$$

$$I = 50\ kg\ m^2$$

(ii) KE initially

$$= \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}(50)(20\pi)^2$$

$$= \frac{1}{2}(50)400\pi^2$$

$$= 10000\pi^2 J$$

which is the loss in KE

Question 3

(a) horizontal in direction BA

(b) Resolving at C, in a vertical direction

$$T_{CB} \cos 60 = 2$$

$$T_{CB} = 4N$$

which is in tension

(c) (i) Resolving vertically over the whole system

$$F_{D\uparrow} = 2$$

(ii) Taking moments about A

$$F_{D\rightarrow}(1 \cos 60) = 2 \times 2(1 \sin 60)$$

$$F_{D\rightarrow} = \frac{4 \sin 60}{\cos 60}$$

$$= 8 \sin 60 = 4\sqrt{3}N$$

Question 4

(a) (i) Resolving vertically upwards

$$2 \cos 30 - 4 \cos 30$$

$$-2\frac{\sqrt{3}}{2}$$

$$-\sqrt{3}$$

Resolving horizontally, to the right

$$3 + 1 + 2 \cos 60 - 4 \cos 60$$

$$4 - 2 \times \frac{1}{2}$$

$$3$$

Resultant R is given by

$$R^2 = (\sqrt{3})^2 + 3^2 = 12$$

$$R = \sqrt{12} = 2\sqrt{3}\ N$$

(ii) Resultant is in direction θ given by

$$\tan^{-1}\theta = \frac{-\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}$$

$$\theta = -30^\circ, \quad \text{i.e. 30 degrees from horizontal}$$

(b) (i) Force required is of same magnitude as resultant found in (a), but in directly opposite direction

(ii) Taking moments about O, clockwise

$$3 \times (1 \cos 30) + 4(\cos 30 \times 1)$$

$$= 7 \cos 30$$

$$= \frac{7\sqrt{3}}{2}$$

so couple applied is of this magnitude, but counter-clockwise

Question 5

(a)

	Mass	C of G from O	Moment of C of G
Cylinder	$\rho \times \pi a^2 \times 2a$	a	$2\pi\rho a^4$
Hemisphere	$k\rho \times \frac{1}{2} \times \frac{4}{3}\pi a^3 = k\rho \times \frac{2}{3}\pi a^3$	$2a + \frac{3}{8}a$	$k\rho\left(\frac{19}{12}\pi a^4\right)$

Total mass

$$= (\rho \times \pi a^2 \times 2a) + \left(k\rho \times \frac{2}{3}\pi a^3\right)$$

$$\rho\pi a^3\left(2 + \frac{4k}{6}\right)$$

$$\rho\pi a^3\left(2 + \frac{2k}{3}\right)$$

Now

$$M\bar{x} = \sum m_i \bar{x}_i$$

$$\rho\pi a^3\left(2 + \frac{2k}{3}\right)\bar{x} = 2\pi\rho a^4 + k\rho\left(\frac{19}{12}\pi a^4\right)$$

$$\left(2 + \frac{2k}{3}\right)\bar{x} = 2a + k\left(\frac{19}{12}a\right)$$

$$\frac{1}{3}(6+2k)\bar{x} = \frac{1}{12}(24a + 19ka)$$

$$\bar{x} = \frac{(24a+19ka)}{4(6+2k)}$$

(b)

$$\frac{(24a+19ka)}{4(6+2k)} > 2a$$

$$24a + 19ka > 2a \times 4(6+2k)$$

$$24 + 19k > 8(6+2k)$$

$$24 + 19k > 48 + 16k$$

$$3k > 24$$

$$k > 8$$

Question 6

(a) Consider an element with length δx at a distance x from one end

The mass of element $= \rho\delta x$ (where ρ is the density per unit length)

M of I of element about one end $= x^2\rho\delta x$

Therefore, M of I of rod is given by

$$I = \rho\int_0^{2a} x^2 dx$$

$$= \rho\left[\frac{x^3}{3}\right]_0^{2a}$$

$$= \frac{8}{3}\rho a^3$$

Since mass (m) of the rod $= 2a\rho$

$$I = \frac{4}{3}ma^2$$

(b) Using formula above

$$I = \frac{4}{3}(8m)a^2$$

$$\frac{32ma^2}{3}$$

(c) By modelling powder as a particle, we can use

$$I = mr^2$$

$$= (8m) \times (2a)^2$$

$$= 32ma^2$$

(d) Initial angular momentum

$$= \frac{32ma^2}{3}\omega_i$$

Final angular momentum

$$= 32ma^2\omega_f$$

Using Conservation of Angular Momentum

$$\frac{32ma^2}{3}\omega_i = 32ma^2\omega_f$$

$$\omega_f = \frac{\omega_i}{3}$$

Mechanics 4 - MAM4, 26 June 2002

Question 1

(a) (i)

$$\frac{dv}{dt} = kt^2 + 10t$$

$$\int dv = \int (kt^2 + 10t).dt$$

$$v = \frac{kt^3}{3} + \frac{10t^2}{2} + c$$

At t=0, v=0

$$0 = c$$

therefore

$$v = \frac{kt^3}{3} + 5t^2$$

(ii) At t=2, v = 25 ms^{-1}

$$25 = \frac{8}{3}k + 20$$

$$k = \frac{5 \times 3}{8} = \frac{15}{8}$$

(b)

$$s = \int_0^2 \left(\frac{5}{8}t^3 + 5t^2\right) dt$$

$$= \left[\frac{5}{32}t^4 + \frac{5}{3}t^3\right]_0^2$$

$$= \frac{5}{32}(16) + \frac{5}{3}(8) - (0)$$

$$= 15\frac{5}{6}m$$

Question 2

(a) Rate of change of momentum in transverse direction is zero

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

$$r^2\dot{\theta} = a$$

$$\dot{\theta} = \frac{a}{r^2}$$

(b)

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{km}{r^2}$$

$$\ddot{r} = r\left(\frac{a}{r^2}\right)^2 - \frac{k}{r^2}$$

$$\ddot{r} = \frac{a^2}{r^3} - \frac{k}{r^2}$$

(c) Multiply expression from (b) by $\dot{r}$

$$\dot{r}\ddot{r} = \dot{r}\frac{a^2}{r^3} - \dot{r}\frac{k}{r^2}$$

Integrating

$$\int \dot{r}\ddot{r}dt = \int \frac{a^2}{r^3}dr - \int \frac{k}{r^2}dr$$

$$\frac{1}{2}\dot{r}^2 = -\frac{a^2}{2r^2} + \frac{k}{r} + c$$

When $r = \frac{a^2}{k}, \dot{r} = \frac{k}{2a}$

$$\frac{1}{2}\left(\frac{k}{2a}\right)^2 = -\frac{a^2k^2}{2(a^2)^2} + \frac{k^2}{a^2} + c$$

$$\Rightarrow c = \frac{k^2}{a^2}\left(\frac{1}{8} + \frac{1}{2} - 1\right) = -\frac{3k^2}{8a^2}$$

so

$$\dot{r}^2 = -\frac{a^2}{r^2} + \frac{2k}{r} - \frac{3k^2}{4a^2}$$

Question 3

Here we are dealing with ballistic rockets

(a)

$$\ddot{r} = -\frac{GM}{r^2}$$

$$= -\frac{6.7 \times 10^{-11} \times 6.0 \times 10^{24}}{x^2}$$

$$= -\frac{4.02 \times 10^{14}}{x^2}ms^{-1}$$

(b)

$$v\frac{dv}{dx} = -\frac{4.02 \times 10^{14}}{x^2}$$

$$\int vdv = -4.02 \times 10^{14} \int \frac{dx}{x^2}$$

$$\frac{1}{2}v^2 = \frac{4.02 \times 10^{14}}{x} + c$$

When $x = 6.4 \times 10^6$, $v = u$

$$\frac{1}{2}u^2 = -\frac{4.02 \times 10^{14}}{6.4 \times 10^6} + c$$

$$c = \frac{1}{2}u^2 - \frac{4.02 \times 10^{14}}{6.4 \times 10^6}$$

expression becomes

$$\frac{1}{2}v^2 = \frac{4.02 \times 10^{14}}{x} + \frac{1}{2}u^2 - \frac{4.02 \times 10^{14}}{6.4 \times 10^6}$$

Inserting $x = (2 \times 10^6 + 6.4 \times 10^6)$ and $v = 0$

$$0 = \frac{4.02 \times 10^{14}}{8.4 \times 10^6} + \frac{1}{2}u^2 - \frac{4.02 \times 10^{14}}{6.4 \times 10^6}$$

$$u = 5.47 \times 10^3 \ ms^{-1}$$

Question 4

(a)

$$T = \lambda x$$
$$mg = \frac{2mg}{l}x$$
$$x = \frac{l}{2}$$

which is the equilibrium extension

(b)

$$mg - T - 2mv\sqrt{\frac{g}{l}} = m\ddot{x}$$
$$mg - \frac{2mg}{l}\left(x + \frac{l}{2}\right) - 2m\dot{x}\sqrt{\frac{g}{l}} = m\ddot{x}$$
$$\ddot{x} + 2\sqrt{\frac{g}{l}}\dot{x} + \frac{2g}{l}x = 0$$

(c) Auxiliary Equation

$$m^2 + 2\sqrt{\frac{g}{l}}m + \frac{2g}{l} = 0$$

which has roots

$$\frac{-2\sqrt{\frac{g}{l}} \pm \sqrt{(2\sqrt{\frac{g}{l}})^2 - 4\left(\frac{2g}{l}\right)}}{2}$$
$$-\sqrt{\frac{g}{l}} \pm \frac{\sqrt{\frac{4g}{l} - \frac{8g}{l}}}{2}$$
$$-\sqrt{\frac{g}{l}} \pm i\sqrt{\frac{g}{l}}$$

so General Solution is

$$x = e^{-\sqrt{\frac{g}{l}}t}\left(A\sin\sqrt{\frac{g}{l}}t + B\cos\sqrt{\frac{g}{l}}t\right)$$

when $t = 0,\ x = \frac{l}{2}$

$$\Rightarrow \frac{l}{2} = B$$

differentiating

$$\dot{x} = e^{-\sqrt{\frac{g}{l}}t}\left(\sqrt{\frac{g}{l}}A\cos\sqrt{\frac{g}{l}}t - \sqrt{\frac{g}{l}}B\sin\sqrt{\frac{g}{l}}t\right)$$
$$+\left(A\sin\sqrt{\frac{g}{l}}t + B\cos\sqrt{\frac{g}{l}}t\right)\left(-\sqrt{\frac{g}{l}}e^{-\sqrt{\frac{g}{l}}t}\right)$$

When $t = 0,\ \dot{x} = 0$

$$0 = \sqrt{\frac{g}{l}}A + B\left(-\sqrt{\frac{g}{l}}\right)$$
$$0 = \sqrt{\frac{g}{l}}A + \frac{l}{2}\left(-\sqrt{\frac{g}{l}}\right)$$
$$A = \frac{l}{2}$$

Particular Solution is

$$x = e^{-\sqrt{\frac{g}{l}}t}\frac{l}{2}\left(\sin\sqrt{\frac{g}{l}}t + \cos\sqrt{\frac{g}{l}}t\right)$$

Question 5

(a)

$$m = \rho\frac{4}{3}\pi r^3$$
$$\frac{dm}{dt} = \frac{4}{3}\rho\pi 3r^2\frac{dr}{dt}$$

Given that $\frac{dr}{dt} = \lambda r$

$$\frac{dm}{dt} = 4\rho\pi r^3\lambda$$
$$= 3m\lambda$$

(b) Rate of change of momentum = Force

$$\frac{d}{dt}(mv) = mg$$
$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$
$$m\frac{dv}{dt} + v(3m\lambda) = mg$$
$$\frac{dv}{dt} = g - 3\lambda v$$

(c)

$$\int \frac{dv}{(g - 3\lambda v)} = \int dt$$
$$-\frac{1}{3\lambda}\ln(g - 3\lambda v) = t + c$$

When t=0, v=u

$$-\frac{1}{3\lambda}\ln(g - 3\lambda u) = c$$

expression becomes

$$-\frac{1}{3\lambda}\ln(g - 3\lambda v) = t - \frac{1}{3\lambda}\ln(g - 3\lambda u)$$
$$\ln(g - 3\lambda v) = -3\lambda t + \ln(g - 3\lambda u)$$
$$g - 3\lambda v = e^{(-3\lambda t + \ln(g - 3\lambda u))}$$
$$3\lambda v = g - e^{-3\lambda t}e^{\ln(g - 3\lambda u)}$$
$$v = \frac{g}{3\lambda} - \frac{e^{-3\lambda t}(g - 3\lambda u)}{3\lambda}$$

(d) Last term above

$$= \frac{(g - 3\lambda u)}{3\lambda e^{3\lambda t}}$$

As $t \to \infty,\ \ e^{3\lambda t} \to \infty$ and whole term $\to 0$, so

$$v = \frac{g}{3\lambda}$$

as $t \to \infty$