AQA Board. **GCE Maths (Specification A).**

Model Answers

January 2004

Note comment on paper about stating final answers to 3 significant figures, in general

Pure 1 - MAP1, 14 January 2004

Question 1

(a)

$$\int x^{\frac{1}{2}}\ dx = \frac{2}{3}x^{\frac{3}{2}} + c$$

(b)

$$\int_0^2 x^{\frac{1}{2}}\ dx$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2$$

$$= \frac{2}{3}2^{\frac{3}{2}} = \frac{2}{3}2\sqrt{2} = \frac{4}{3}\sqrt{2}$$

Question 2

$$u_n = 2 \times 3^n$$

(a)

$$u_1 = 6$$

$$u_2 = 18$$

(b) 3

(c) The general sum is given by

$$S_n = a\frac{(1-r^n)}{(1-r)}$$

and the sum of the first 10 terms is given by

$$S_{10} = 6\frac{(1-3^{10})}{(1-3)}$$

$$= -3(1-3^{10}) = 3(3^{10}-1)$$

Could also have stated the general sum above as

$$S_n = a\frac{(r^n-1)}{(r-1)}$$

Question 3

(a) Area of a sector is

$$\frac{1}{2}\theta r^2$$

Therefore the area of ADE is

$$\frac{1}{2}\theta(5)^2$$

$$= \frac{25\theta}{2}$$

(b) (i) Area of ABCD = $25cm^2$.
Since the Area of ADE = $\frac{1}{4}\times$ Area of ABCD

$$\frac{25\theta}{2} = \frac{25}{4}$$

Therefore

$$\theta = \frac{1}{2}$$

(ii) Length of ED = $r\theta = 5\ \times\frac{1}{2} = 2.5$
Other four sides are 5cm long, so total perimeter = $(5 \times 4) + 2.5 = 22.5$ cm

Question 4

(a) (i) General term is given by

$$u_n = a + (n-1)d$$

So second term is

$$u_2 = 100 + (1)2 = 102$$

and third term is

$$u_3 = 100 + (2)2 = 104$$

(Could have just derived these answers by mental arithmetic)

(ii) From the formula for general term

$$200 = 100 + (n-1) \times 2$$

$$100 = (n-1) \times 2$$

$$(n-1) = 50 \Rightarrow n = 51$$

(b) From above, the no. of layers is 51
Formula for sum of an arithmetic series is

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Here

$$S_{51} = \frac{51}{2}(200+100) = 7650\ mm$$

Question 5

$$y = e^{2x} + 2x^{-1}$$

(a)

$$\frac{dy}{dx} = 2e^{2x} - 2x^{-2}$$

(b) At stationary point

$$\frac{dy}{dx} = 0$$

so

$$2e^{2x} - 2x^{-2} = 0 \Rightarrow e^{2x} = x^{-2} \Rightarrow x^2e^{2x} = 1$$

(c) Rearranging

$$x^2e^{2x} = 1$$

gives

$$(xe^x)^2 = 1$$

so

$$xe^x = 1 \qquad \textit{when } x > 0, \textit{ as here}$$

Take the log of both sides

$$\ln(xe^x) = \ln(1)$$

$$\ln(x) + \ln(e^x) = 0$$

$$\ln x + x = 0$$

(d) When x = 0.5

$$\ln x + x = \ln 0.5 + 0.5 = -0.193$$

When x = 0.6

$$\ln x + x = \ln 0.6 + 0.6 = 0.0892$$

Because the equation changes sign in the interval, it has a root in the interval

(e)

$$\int \left(e^{2x} + 2x^{-1}\right)\, dx$$

$$= \frac{e^{2x}}{2} + 2\ln x + c$$

Question 6

(a) (i)

$$fg(x) = f(x-1) = \sqrt{x-1}$$

$$gf(x) = g(\sqrt{x}) = \sqrt{x} - 1$$

(ii) If x= 1

$$fg(x) = 0$$

$$gf(x) = 0$$

(b) (i) translation of +1 in the x direction

(ii) Within the domain

$$1 \le x \le 5$$

the range of h is

$$0 \le y \le 2$$

(iii) For the inverse function, the range would be

$$1 \le x \le 5$$

and the domain would be

$$0 \le y \le 2$$

(iv) Let

$$y = \sqrt{x-1}$$

Therefore

$$y^2 = x - 1$$

$$x = y^2 + 1$$

which defines $h^{-1}(y)$

$h^{-1}(x)$ involves shifting x and y around to give the definition

$$h^{-1}(x) = x^2 + 1$$

Question 7

(a)

$$\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

(b)

$$3\sin^2 x = \cos^2 x$$

$$3(1 - \cos^2 x) = \cos^2 x$$

$$3 - 3\cos^2 x = \cos^2 x$$

$$4\cos^2 x = 3$$

$$\cos^2 x = \frac{3}{4}$$

$$\cos x = \frac{\sqrt{3}}{2}$$

which produces the general solution

$$x = 2n\pi \pm \frac{\pi}{6}, \qquad n = 0, 1, 2, ...$$

Within 0 and 2π the required solutions will be

$$\frac{\pi}{6}, \quad \frac{11\pi}{6}$$

Pure 2 - MAP2, 19 January 2004

Question 1

(a) (i) $\alpha\beta = \frac{1}{2}$

(ii) $\alpha + \beta = 3$

(b) (i)

$$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = 2$$

(ii)

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{3}{\frac{1}{2}} = 6$$

(c)

$$x^2 - 6x + 2 = 0$$

Hint : if uncertain about the formulas used in part (a), you can work them out from first principles easily. Take (x-a)(x-b) where a and b are the roots, get rid of the brackets and compare with general form of a quadratic

Question 2

$$x^2 + y^2 - 4x + 4y - 12 = 0$$

(a) (i)

$$(2, -2)$$

(ii) Radius (r) is given by

$$r^2 = (-2)^2 + 2^2 - (-12)$$

$$r^2 = 4 + 4 + 12 = 20$$

Giving

$$r = \sqrt{20}$$

Alternatively, you could use the coordinates from (a)(i) to form

$$(x-2)^2 + (y+2)^2 = 20$$

where the RHS is equal to r^2

(b) When the circle crosses the x-axis, then y=0 and equation becomes

$$x^2 - 4x - 12 = 0$$

which factorizes to

$$(x+2)(x-6) = 0$$

Giving x= -2 and x= 6

Therefore required coordinates are

$$(-2, 0) \ \ and \ \ (6, 0)$$

(c) Differentiating the circle equation

$$2x + 2y\frac{dy}{dx} - 4 + 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y+4) = -2x + 4$$

$$\frac{dy}{dx} = \frac{(-2x+4)}{(2y+4)}$$

at (4,2)

$$\frac{dy}{dx} = \frac{-4}{8} = -\frac{1}{2}$$

Therefore equation of tangent is given by

$$y - y_0 = -\frac{1}{2}(x - x_0)$$

With $y_0 = 2$ and $x_0 = 4$

$$y - 2 = -\frac{1}{2}(x-4)$$

$$y = -\frac{1}{2}x + 4$$

Question 3

(a) 1.176 rads

(b) Let

$$10\sin\theta + 24\cos\theta = R\sin(\theta + \alpha)$$

$$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

equating LHS and RHS gives

$$10 = R\cos\alpha \qquad (1)$$

$$24 = R\sin\alpha \qquad (2)$$

Dividing (8) by (7) gives

$$\tan\alpha = 24/10 = 2.4$$

$$\alpha = 1.176\, rads \;\; (from\ above)$$

Squaring both (7) and (8) and adding, gives

$$R^2 = 10^2 + 24^2 = 676$$

$$R = 26$$

(Note - making use of $\sin^2 x + \cos^2 x = 1$*)*

Therefore, the required form is

$$26 \sin(\theta + 1.176)$$

(c) (i) We have converted the expression into a sine form. The maximum value of a sine is 1, so when the sine above equals 1, this will give the maximum value of the expression, i.e. 26

(ii) Let

$$\sin(\theta + 1.176) = 1$$

$$\theta + 1.176 = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{2} - 1.176 = 0.395\, rads$$

Question 4

(a)

$$y = \ln(x^2 + 9)$$

Using the chain rule,

let $u = x^2 + 9$	$y = \ln u$
$\frac{du}{dx} = 2x$	$\frac{dy}{du} = \frac{1}{u}$

Since

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times 2x = \frac{2x}{u} = \frac{2x}{(x^2+9)}$$

(b)

$$\int_0^3 \frac{x}{x^2+9} dx = \frac{1}{2} \int_0^3 \frac{2x}{x^2+9} dx$$

From part (a), we know this integral, so it becomes

$$\frac{1}{2} \left[\ln(x^2+9)\right]_0^3$$

$$= \frac{1}{2}\{\ln 18 - \ln 9\}$$

$$= \frac{1}{2} \ln \frac{18}{9} = \frac{1}{2} \ln 2$$

(c)

$$\int_0^3 \frac{x+1}{x^2+9} dx = \int_0^3 \frac{x}{x^2+9} dx + \int_0^3 \frac{1}{x^2+9} dx$$

so need to calculate the 'extra' integral on the right

$$\int_0^3 \frac{1}{x^2+9} dx = \frac{1}{3} \left[tan^{-1} \frac{x}{3}\right]_0^3$$

$$= \frac{1}{3}\{tan^{-1} 1 - \tan^{-1} 0\}$$

$$= \frac{1}{3} . \frac{\pi}{4} - 0 = \frac{\pi}{12}$$

Adding this expression on to the result from (b) gives the required result.

$$\frac{1}{2} \ln 2 + \frac{\pi}{12}$$

Question 5

(a) Trapezium Rule is

$$Integral = h \left\{ \frac{1}{2}(f_0 + f_n) + \sum_{(n-1)}^{i=1} f_i \right\}$$

where h is the interval

In this case, with four strips, we will have

$$f_0 = 0; f_1 = 0.6082;$$

$$f_2 = 1.38629; f_3 = 2.29073; f_4 = 3.29584 \;\; and \;\; h = \frac{1}{2}$$

So integral from trapezium rule

$$= \frac{1}{2} \left\{ \frac{1}{2}(0 + 3.29584) + (0.6082 + 1.38629 + 2.29073) \right\}$$

$$= 2.97$$

(b) (i) If

$$y = 2x^2 \ln x - x^2$$

$$\frac{dy}{dx} = 2(2x \ln x + x^2 . \frac{1}{x}) - 2x$$

$$= 4x \ln x + 2x - 2x$$

$$= 4x \ln x$$

(ii)

$$\int_1^3 x \ln x\, dx = \frac{1}{4} \int_1^3 4x \ln x\, dx$$

By using the result from (i), this integral becomes

$$\frac{1}{4} \left[2x^2 \ln x - x^2\right]_1^3$$

$$= \frac{1}{4} \{18 \ln 3 - 9) - (2 \ln 1 - 1)\}$$

$$= \frac{1}{4}\{18 \ln 3 - 9 + 1\}$$

$$= \frac{1}{4}\{18 \ln 3 - 8\}$$

$$= 2.94$$

Question 6

(a) Given

$$\sin x - \frac{1}{2}x = 0$$

When x=1

$$\sin 1 - \frac{1}{2} = 0.841 - 0.5 = 0.341$$

When x=2

$$\sin 2 - \frac{1}{2}(2) = 0.909 - 1 = -0.0907$$

Because the expression changes sign in the interval, there must be a root in the interval itself.

(b) (i)

$$f(x) = \sin x - \frac{1}{2}x$$

Therefore

$$f'(x) = \cos x - \frac{1}{2}$$

(ii) Newton-Raphson iteration is given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

So taking $x_n = 2$

$$x_{n+1} = 2 - \frac{-0.0907}{-0.416 - 0.5}$$

$$x_{n+1} = 2 - \frac{-0.0907}{-0.916}$$

$$x_{n+1} = 1.901 \ \textit{to four sig. figures} \ \approx 1.9$$

(c) (i)

$$\int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x) dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + c$$

(ii) Hence

$$\int_0^{1.9} \sin^2 x \, dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{1.9}$$

$$= \left(0.95 - \frac{1}{4}\sin 3.8\right) - (0)$$

$$= 1.10 \ \textit{to } 3 \ \textit{sig. figs}$$

(d) Volume of revolution is given by

$$V = \pi \int_0^{1.9} \sin^2 x \, dx - \pi \int_0^{1.9} \frac{1}{4}x^2 dx$$

$$= \pi \left(1.10 - \pi \left[\frac{1}{4}.\frac{x^3}{3}\right]_0^{1.9}\right)$$

$$= \pi \left(1.10 - \frac{1}{4}(2.286 - 0)\right)$$

$$\pi(1.10 - 0.572)$$

$$= 1.7 \ \textit{to } 2 \ \textit{sig. figs}$$

Pure 3 - MAP3, 23 January 2004

Question 1

(a) (i)

$$x = 3t^2 \quad y = 6t$$

$$\tfrac{dx}{dt} = 6t \quad \tfrac{dy}{dt} = 6$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 6 \times \frac{1}{6t}$$

$$= \frac{1}{t}$$

(ii) When $t = \frac{1}{2}$, the gradient

$$= \frac{1}{\frac{1}{2}} = 2$$

(b) (i) Since

$$y = 6t$$

$$t = \frac{y}{6}$$

Substitute this in

$$x = 3t^2$$

giving

$$x = 3\left(\frac{y}{6}\right)^2$$

$$x = 3\left(\frac{y^2}{36}\right)$$

$$x = \frac{y^2}{12}$$

(ii)

$$\frac{dx}{dy} = \frac{y}{6}$$

When $t = \frac{1}{2}$, y = 3, so

$$\frac{dx}{dy} = \frac{3}{6} = \frac{1}{2}$$

and

$$\frac{dy}{dx} = 2$$

which is the same answer as for (a)(ii)

Question 2

$$\frac{dy}{dx} = \frac{1}{1+3x^2}$$

Therefore, when x = 1 the gradient

$$= \frac{1}{1+3} = \frac{1}{4}$$

Using

$$y - y_0 = m(x - x_0)$$

with $(x_0, y_0) = (1, 0.5)$ and $x = 1.25$

$$y = 0.5 + \frac{1}{4}(1.25 - 1)$$

$$y = 0.5625$$

At x=1.25, the gradient

$$= \frac{1}{1 + 1.25^3} = 0.3386$$

with $(x_0, y_0) = (1.25, 0.5625)$ and $x = 1.5$

$$y = 0.5625 + 0.3386(1.5 - 1.25)$$

$$y = 0.647$$

Question 3

(a) (i) 50

(ii) 100

(b) Insert population of 75 into the general formula,

$$75 = 100 - 50e^{-\frac{1}{4}t}$$

$$50e^{-\frac{1}{4}t} = 25$$

$$e^{-\frac{1}{4}t} = \frac{1}{2}$$

$$-\frac{1}{4}t = \ln\frac{1}{2}$$

$$t = -4\ln\frac{1}{2} = +4\ln 2 = 2.8$$

Question 4

(a)

$$\frac{8+3x}{(1+3x)(2-x)} = \frac{A}{1+3x} + \frac{B}{2-x}$$

so

$$8+3x = A(2-x) + B(1+3x)$$

Equating coefficients of x

$$3 = -A + 3B$$

Equating numerals

$$8 = 2A + B$$

From (8)

$$B = 8 - 2A$$

substituting this into (7) gives

$$3 = -A + 3(8-2A)$$

$$3 = -A + 24 - 6A$$

$$7A = 21$$

$$\Rightarrow A = 3 \;\; and \;\; B = 2$$

therefore, required form is

$$\frac{8+3x}{(1+3x)(2-x)} = \frac{3}{1+3x} + \frac{2}{2-x}$$

(b)

$$\frac{1}{1+3x} = (1+3x)^{-1}$$

$$= 1 - 3x + \frac{(-1)(-2)(3x)^2}{2!}$$

$$= 1 - 3x + 9x^2$$

(c)

$$\frac{1}{2-x} = (2-x)^{-1}$$

$$= (2)^{-1}\left(1 - \frac{x}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1 + \frac{x}{2} + \frac{(-1)(-2)(-x/2)^2}{2!}\right)$$

$$= \frac{1}{2}(1 + \frac{x}{2} + \frac{x^2}{4})$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8}$$

(d) Multiplying the answers from (b) and (c) together (ignoring all powers greater than x^2), transforms the expression to

$$(8+3x)\left(\frac{1}{2} - \frac{3}{2}x + \frac{9}{2}x^2 + \frac{x}{4} - \frac{3}{4}x^2 + \frac{1}{8}x^2\right)$$

$$= (8+3x)\left(\frac{1}{2} - \frac{5}{4}x + \frac{31}{8}x^2\right)$$

$$= 4 - \frac{40}{4}x + 31x^2 + \frac{3}{2}x - \frac{15}{4}x^2$$

$$= 4 - 8.5x + 27.25x^2$$

(e) In (b), the expansion was valid for $|3x| \le 1$, giving $|x| \le \frac{1}{3}$. So the expansion is valid for

$$-\frac{1}{3} \le x \le \frac{1}{3}$$

Question 5

(a) (i)

$$f(x) = e^{-2x}$$

$$f'(x) = -2e^{-2x}$$

$$f''(x) = 4e^{-2x}$$

(ii) MacLaurin series is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$$

and for the derivatives from (a)

$$f(0) = 1$$

$$f'(0) = -2$$

$$f''(0) = 4$$

giving MacLaurin's expansion for this function as

$$f(x) = 1 - 2x + 2x^2$$

(b) (i)

$$\cos 3x \approx 1 - \frac{(3x)^2}{2}$$

$$\approx 1 - \frac{9x^2}{2}$$

(ii)

$$e^{-2x} = \cos 3x$$

$$1 - 2x + 2x^2 = 1 - \frac{9x^2}{2}$$

$$\frac{13}{2}x^2 - 2x = 0$$

$$x\left(\frac{13}{2}x - 2\right) = 0$$

$x = 0$ does not fall within the required range. If we set

$$\left(\frac{13}{2}x - 2\right) = 0$$

we get

$$x = \frac{4}{13}$$

which is the required answert

Question 6

$$\frac{dv}{dt} = 10 - 5v$$

(a) Rearranging, we get

$$\int dt = \int \frac{dv}{(10-5v)}$$

$$\int dt = -\frac{1}{5}\int \frac{-5\ dv}{(10-5v)}$$

$$t = -\frac{1}{5}\ln(10-5v) + c$$

When t=0, v=0 - substitute these values in above formula

$$c = \frac{1}{5}\ln(10)$$

producing the full equation

$$t = -\frac{1}{5}\ln(10-5v) + \frac{1}{5}\ln(10)$$

$$t = \frac{1}{5}\ln\left(\frac{2}{2-v}\right)$$

(b) Manipulating the above expression

$$5t = \ln\left(\frac{2}{2-v}\right)$$

$$e^{5t} = \frac{2}{2-v}$$

$$2 - v = \frac{2}{e^{5t}}$$

$$v = 2 - \frac{2}{e^{5t}}$$

When t = 0.5s

$$v = 2 - \frac{2}{e^{2.5}}$$

$$v = 1.8 \quad \textit{to 2 sig figs.}$$

Question 7

(a) (i) Distance (r) between A and B is given by

$$r^2 = (3-5)^2 + (-1-3)^2 + (2-(-2))^2$$

$$r^2 = 4 + 16 + 16 = 36$$

Therefore

$$r = 6$$

(ii) The coordinates of the midpoint are given by

$$\left(\frac{3+5}{2}, \frac{-1+3}{2}, \frac{2-2}{2}\right)$$

$$= (4, 1, 0)$$

(b)

$$\vec{CM} = (4-8, 1-(-2), 0-(-1))$$

$$= (-4, 3, 1)$$

$$\vec{AB} = (5-3, 3-(-1), -2-2)$$

$$= (2, 4, -4)$$

The scalar product of $\vec{CM}$ and $\vec{AB}$ is

$$(-4, 3, 1).(2, 4, -4) = -8 + 12 - 4 = 0$$

(c) The equation of a plane is given by

$$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$$

where **a** is the position vector of the perpendicular **n**. Therefore

$$\mathbf{r}.\mathbf{n} = (8, -2, -1).(-4, 3, 1)$$

$$\mathbf{r}.\mathbf{n} = -32 - 6 - 1 = -39$$

Since $\mathbf{r} = (\mathrm{x}, \mathrm{y}, \mathrm{z})$

$$(x, y, z).(-4, 3, 1) = -39$$

$$-4x + 3y + z = -39$$

(d) We require that $\mathbf{r}.\mathbf{n} = 13$, i.e.

$$-4(8+5t) + 3(-2-3t) + (-1+3t) = 13$$

$$-32 - 20t - 6 - 9t - 1 + 3t = 13$$

$$26t = 52 \Rightarrow t = 2$$

Therefore, required co-ordinates are

$$(8+10, -2-6, -1+6)$$

$$(18, -8, 5)$$

Pure 4 - MAP4, 19 January 2004

Question 1

(a) (i)

$$(3+i)^2$$

$$= 9 + 6i + i^2 = 8 + 6i$$

(ii)

$$(2+4i)(3+i) = 6 + 2i + 12i + 4i^2$$

$$2 + 14i$$

(b)

$$z^2 - (2+4i)z + 8i - 6 = 0$$

(i) Substituting $z_1 = 3 + i$ into the LH side of the equation gives

$$(3+i)^2 - (2+4i)(3+i) + 8i - 6$$

Using the results above gives

$$8 + 6i - (2 + 14i) + 8i - 6 = 0$$

so $z_1 = 3 + i$ is a root

(ii) 2+4i

(iii) Non-real roots will occur in conjugate pairs only for quadratics which have **real** coefficients, which is not the case here.

(iv)

$$z_2 = 2 + 4i - (3+i) = 2 + 4i - 3 - i = -1 + 3i$$

(c) (i) Argand Diagram

(ii) Magnitude of $z_1 = 3 + i = \sqrt{9+1} = \sqrt{10}$
Magnitude of $z_2 = -1 + 3i = \sqrt{1+9} = \sqrt{10}$

(iii) arg z_1
$\tan^{-1}\frac{1}{3} = 18.43$ degrees
arg(z_2)
$\tan^{-1}(-3) = -71.57$ degrees, but the argument will be the principal value counter-clockwise from zero, so arg $= 108.43$ degrees.

$$arg\left(\frac{z_2}{z_1}\right) = arg(z_2) - arg(z_1) = 90\ degrees$$

Question 2

$$\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^7 \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^5$$

$$\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\left(\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}\right)$$

$$\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$-\frac{\sqrt{3}}{4} - \frac{3}{4}i - \frac{1}{4}i - \frac{\sqrt{3}}{4}i^2$$

$$= -i$$

Alternatively, from the second line you could expand the brackets producing

$$= \cos\left(\frac{7\pi}{6}\right)\cos\left(\frac{5\pi}{3}\right) + \sin\left(\frac{7\pi}{6}\right)\sin\left(\frac{5\pi}{3}\right)$$

$$+ i\sin\left(\frac{7\pi}{6}\right)\cos\left(\frac{5\pi}{3}\right) - i\cos\left(\frac{7\pi}{6}\right)\sin\left(\frac{5\pi}{3}\right)$$

$$= \cos\left(\frac{7\pi}{6} - \frac{5\pi}{3}\right) + i\sin\left(\frac{7\pi}{6} - \frac{5\pi}{3}\right)$$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$-i$$

Question 3

$$f(n) = n^3 + (n+1)^3 + (n+2)^3$$

(a)

$$f(n+1) = (n+1)^3 + (n+2)^2 + (n+3)^2$$

So

$$f(n+1) - f(n) = (n+3)^3 - n^3$$

$$= \left(n^3 + 3n^2(3) + \frac{(3)(2)}{2!}n(3)^2 + \frac{(3)(2)(1)}{3!}(3)^2\right) - n^3$$

$$= n^3 + 9n^2 + 27n + 9 - n^3$$

$$= 9n^2 + 27n + 9$$

(b) Since the difference between f(n+1) and f(n), as given in (a) **is** divisible by 9, we need to show that for one particular choice of n, the expression f(n) is divisible by 9. If this is so, then the answer from (a) will ensure that any other choice of n will also be divisible by 9.

Choose n= 1

$$f(1) = 1 + 2^3 + 3^3 = 9 + 3^3$$

which is indeed divisible by 9

Question 4

(a)

$$y = \sinh^{-1} x$$

Therefore

$$x = \sinh y$$
$$\frac{dx}{dy} = \cosh y$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

since

$$\cosh^2 y - \sinh^2 y = 1$$

then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\sinh^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$x = \sinh y$$
$$\frac{dx}{dy} = \cosh y$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

since

$$\cosh^2 y - \sinh^2 y = 1$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\sinh^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

Alternatively

$$\sinh y = x$$

$$\cosh y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

(b) (i) For C_1

$$y = \sinh x$$
$$\frac{dy}{dx} = \cosh x$$

At x= 0, the gradient = $\cosh 0 = 1$

From (a), the gradient of C_2 at x=0,

$$= \frac{1}{\sqrt{1+0}} = 1$$

(ii) C_1 : $\cosh x$ has a minimum value at x=0, and so for all other values the gradient of C_1 will be greater than 1.

C_2 : values of x differing from 0 will be squared and have the effect of making the denominator in $\frac{dy}{dx}$ greater than 1, thereby making the gradient less than 1.

(iii) Sketch

Question 5

$$y = \ln(1 - x^2)$$

(a)

$$\frac{dy}{dx} = \frac{-2x}{1-x^2}$$

so

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-2x}{1-x^2}\right)^2$$
$$= \frac{(1-x^2)^2}{(1-x^2)^2} + \frac{(-2x)^2}{(1-x^2)^2}$$
$$= \frac{1 + x^4 - 2x^2 + 4x^2}{(1-x^2)^2}$$
$$= \frac{x^4 + 2x^2 + 1}{(1-x^2)^2}$$
$$= \frac{(1+x^2)^2}{(1-x^2)^2}$$
$$= \left(\frac{(1+x^2)}{(1-x^2)}\right)^2$$

(b) Using

$$ds = \sqrt{1 + (\dot{y})^2}\, dx$$
$$ds = \sqrt{\left(\frac{(1+x^2)}{(1-x^2)}\right)^2}\, dx$$
$$ds = \frac{(1+x^2)}{(1-x^2)}\, dx$$
$$s = \int_0^p \left(\frac{2}{1-x^2} - 1\right) dx$$
$$s = \left[2\tanh^{-1} x - x\right]_0^p$$
$$s = (2\tanh^{-1} p - p) - (0)$$

i.e.

$$s = (2\tanh^{-1} p - p)$$

Question 6

(a) (i)

$$z = 2e^{\frac{1}{4}\pi i}$$

so

$$z^4 = \left(2e^{\frac{1}{4}\pi i}\right)^4$$
$$= 2^4 e^{\pi i} = 16(-1) = -16$$

so z is a root of the equation

(ii)

$$z^4 = -16$$

or

$$z^4 = 16(-1) = 16e^{i\pi + in2\pi}$$

therefore

$$z = \left(16e^{i\pi(1+2n)}\right)^{\frac{1}{4}}$$
$$= 16^{\frac{1}{4}} e^{\frac{i\pi(1+2n)}{4}}$$

so, the four roots between $-\pi < \theta \leq \pi$ are

$$z = \quad 2e^{\frac{i/\pi}{4}}$$
$$2e^{\frac{i3/\pi}{4}}$$
$$2e^{\frac{-i3/\pi}{4}}$$
$$2e^{\frac{-i/\pi}{4}}$$

(iii) Argand Diagram

(b) (i)

$$\left(z - 2e^{\frac{1}{4}\pi i}\right)\left(z - 2e^{-\frac{1}{4}\pi i}\right)$$

$$= z^2 - 2ze^{-\frac{1}{4}\pi i} - 2ze^{\frac{1}{4}\pi i} + 4$$
$$= z^2 - 2z\left(e^{-\frac{1}{4}\pi i} + e^{\frac{1}{4}\pi i}\right) + 4$$

Now

$$e^{-\frac{1}{4}\pi i} + e^{\frac{1}{4}\pi i}$$
$$= \cos\left(\frac{1}{4}\pi i\right) - \sin\left(\frac{1}{4}\pi i\right) + \cos\left(\frac{1}{4}\pi i\right) + \sin\left(\frac{1}{4}\pi i\right)$$
$$= 2\cos\left(\frac{1}{4}\pi i\right) = 2\frac{1}{\sqrt{2}} = \sqrt{2}$$

Inserting back into equation gives

$$= z^2 - 2\sqrt{2}z + 4$$

(ii) Since (a) has produced the product of two factors of $z^4 = -16$, the other **quadratic** factor will be produced by multiplying the other two factors, i.e.

$$\left(z - 2e^{\frac{i3\pi}{4}}\right)\left(z - 2e^{\frac{-i3\pi}{4}}\right)$$

$$z^2 - 2z\left(e^{\frac{i3\pi}{4}} + e^{-\frac{i3\pi}{4}}\right) + 4$$

which by applying the same logic as before becomes

$$(z^2 + 2\sqrt{2}z + 4)$$

So required product is

$$(z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)$$

(This could have been 'guessed at' by reversing the sign in the answer from (a) and doing a quick multiplication to check)
To clarify the theory

$$z^4 = (z - a)(z - b)(z - c)(z - d)$$

where a,b,c,d are the roots. These brackets can be reduced by multiplication to a product of two brackets, just as in the final answer to (b)

Pure 5 - MAP5, 16 January 2004

Question 1

$$\frac{d^2y}{dx^2} = x^2 + y^2$$

and

$$y_{r+1} \approx 2y_r - y_{r-1} + h^2 y_r''$$

so

$$y(1.2) \approx 2 \times 2.08 - 2 + (0.1)^2(1.1^2 + 2.08^2)$$

$$= 2.22$$

Question 2

(a)

$$\int_0^a \frac{x}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{1}{2}\int_0^a \frac{-2x}{\sqrt{1-x^2}}\, dx$$

this integrates to

$$= \left[-\sqrt{1-x^2}\right]_0^a$$

$$= \left(-\sqrt{1-a^2}\right) - (-1)$$

$$= -\sqrt{1-a^2} + 1$$

This integration has been carried out by recognizing the integral to the result of differentiation by the chain rule. To be more specific when we have an integral of the form

$$(g(x))^n\; f(x)\, dx$$

where $f(x)$ is the derivative of $g(x)$ then $(g(x))^n$ can be integrating "using the normal rules", i.e. raise the index and divide through by this new index. Here we have

$$\int \frac{1}{(1-x^2)^{\frac{1}{2}}}(-2x)\, dx$$

If you are uncertain about this procedure, you can use a substitution to solve the integral, and this could also help you to understand the logic behind the procedure outlined in the previous paragraph. Let $u = 1 - x^2$, giving also $\frac{du}{dx} = -2x$.

Alternatively, you can substitute a trigonometrical function. An appropriate substitution for $\sqrt{a^2 - x^2}$ is

$$x = a\sin\theta \quad \textit{and} \quad \frac{dx}{d\theta} = a\cos\theta$$

(b) The denominator in the integral is zero when a=1

(c) From the answer from (a)

$$-\sqrt{1-a^2} + 1$$

as $a \to 0$, this tends to

$$-\sqrt{1-1^2} + 1 = 1$$

Question 3

(a)

$$dA = \frac{1}{2}r^2 d\theta$$

so, given $r = e^{k\theta}$

$$Area = \frac{1}{2}\int_{\theta_1}^{\theta_2} e^{2k\theta}\, d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2k}e^{2k\theta}\right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{4k}\left(e^{2k\theta_2} - e^{2k\theta_1}\right)$$

$$= \frac{1}{4k}\left(r_2^2 - r_1^2\right)$$

(b) (i) At point of intersection

$$e^\theta = 2$$

$$\theta = \ln 2$$

so required coordinates are

$$(2, \ln 2)$$

(ii) Area of R =

$$\frac{1}{2}\int_0^{\ln 2}\left((2)^2 - \left(e^\theta\right)^2\right)\, d\theta$$

$$= \frac{1}{2}\left[4\theta - \frac{1}{2}e^{2\theta}\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left\{\left(4\ln 2 - \frac{1}{2}e^{2\ln 2}\right) - (0 - \frac{1}{2})\right\}$$

$$= \frac{1}{2}\left\{4\ln 2 - 2 + \frac{1}{2}\right\}$$

$$= 2\ln 2 - \frac{3}{4}$$

Question 4

(a) (i)

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\frac{1}{\cos x} = (\cos x)^{-1} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^{-1}$$

$$1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2$$

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}$$

(ii)

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right)$$

$$= x - \frac{x^3}{6} + \frac{x^3}{2} + \frac{5x^5}{24} - \frac{x^5}{12} + \frac{x^5}{120}$$

$$= x + \frac{2x^3}{6} + \frac{16x^5}{120}$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15}$$

(b)

$$\lim_{x\to 0}\left(\frac{\tan 2x - 2x}{\tan x - x}\right)$$

expression inside bracket becomes (using result from (a)(ii)

$$\frac{2x + \frac{8x^3}{3} + \frac{64x^5}{15} - 2x}{x + \frac{x^3}{3} + \frac{2x^5}{15} - x}$$

$$= \frac{\frac{8x^3}{3} + \frac{64x^5}{15}}{\frac{x^3}{3} + \frac{2x^5}{15}}$$

$$= \frac{\frac{8}{3} + O(x^2)}{\frac{1}{3} + O(x^2)}$$

(Note :using $O(x^2)$ because both series have actaully been truncated)

which as $x \to 0$

$$= 8$$

Question 5

(a) Given

$$y = ax^2 + bx$$

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

therefore, substituting these values into

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

produces

$$2a + 2ax + b = x$$

so, equating coefficients of x

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

and equating constants

$$2a + b = 0 \Rightarrow b = -2a \Rightarrow b = -1$$

(b) The auxiliary eqn. is

$$m^2 + m = 0$$

$$m(m+1) = 0$$

so

$$m = 0 \;\; or \;\; m = -1$$

producing the complementary function

$$y = A + Be^{-x}$$

and the general solution

$$y = A + Be^{-x} + \frac{1}{2}x^2 - x$$

when $x = 0, y = 1$, so

$$1 = A + B$$

differentiating G.S.

$$\frac{dy}{dx} = -Be^{-x} + x - 1$$

when $x = 0$, $\frac{dy}{dx} = 3$, so

$$3 = -B - 1$$

$$\Rightarrow B = -4$$

and

$$A = 1 - B \Rightarrow A = 5$$

so General Solution is

$$y = 5 - 4e^{-x} + \frac{1}{2}x^2 - x$$

Question 6

(a) Given $y(1) = 1$, and $h = 0.1$

$$k_1 = 0.1 \times \frac{1+1}{1.1} = 0.2$$

$$k_2 = 0.1 \times \frac{(1+0.1)^3 + (1+0.2)^3}{(1.1)(1+0.2)^3} = 0.193118686$$

$$y(1.1) \approx 1 + \frac{1}{2}(0.2 + 0.193118686) = 1.1966$$

(b) (i) Equate the two formulas for $\frac{dy}{dx}$

$$u + x\frac{du}{dx} = \frac{x^3 + y^3}{xy^2}$$

$$u + x\frac{du}{dx} = \frac{x^3 + u^3x^3}{xu^2x^2} = \frac{1}{u^2} + u$$

so

$$x\frac{du}{dx} = \frac{1}{u^2}$$

(ii)

$$x\frac{du}{dx} = \frac{1}{u^2}$$

$$\frac{1}{x}dx = u^2\ du$$

$$\ln x = \frac{u^3}{3} + c$$

$$\ln x = \frac{y^3}{3x^3} + c$$

Given y(1) = 1

$$\ln 1 = \frac{1^3}{3 \times 1^3} + c \Rightarrow -\frac{1}{3}$$

so

$$\ln x = \frac{y^3}{3x^3} - \frac{1}{3}$$

$$y^3 = 3x^3\left(\ln x + \frac{1}{3}\right)$$

$$y = x\sqrt[3]{3\ln x + 1}$$

(iii)

$$y(1.1) = 1.1\sqrt[3]{3\ln 1.1 + 1} = 1.1962$$

Pure 6 - MAP6, 16 January 2004

Question 1

(a) (i)

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -2 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= (0-(-6))\mathbf{i} - (0-(-4))\mathbf{j} + (9-8)\mathbf{k}$$

$$= 6\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

(ii)

$$(\mathbf{a}\times\mathbf{b}).\mathbf{c}$$

$$= (6\mathbf{i} - 4\mathbf{j} + \mathbf{k}).(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

$$6 - 8 + 2 = 0$$

(b) O, A, B and C are co-planar

Question 2

(a)

$$\begin{vmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 2(-2-0) - 3(2-0) + (-2)(-1-0)$$

$$-4 - 6 + 2 = -8$$

(b) Since the determinant of vectors is not zero, as shown in (a), the vectors are **not** linearly dependent.

(c) We require

$$l\mathbf{u} + m\mathbf{v} + \mathrm{n}\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

i.e.

$$\begin{aligned} 2l + 3m - 2n &= 0 \quad (3) \\ l - m &= 3 \quad (4) \\ -m + 2n &= -2 \quad (5) \end{aligned}$$

Substituting (5) into (7)

$$2l + 2m + 2 = 0 \quad (6)$$

Twice times (8) and subtract from (6)

$$4m = -8 \Rightarrow m = -2$$

Therefore

$$2 + 2n = -2 \Rightarrow n = -2$$

and

$$l = m + 3 = -2 + 3 = 1$$

Question 3

(a) A rotation of -90 degrees about the y -axis

(Note : Since there is a 1 on the diagonal, on the second row, this implies that it represents a rotation about the y-axis. Examination of the corner values correspond to an angle of -90 degrees.)

(b) (i)

$$A \to (0,0,1)$$

$$B \to (0,-1,0)$$

$$C \to ((1,0,0)$$

(Note : Just calculate the new positions by inspection)

(ii) *(Note : the way to do this is to first state the effect of the transformation on the unit vectors of the basis, which we have done in (i), i.e. the effect on the unit vector in x-direction is (0,0,1) etc. The required matrix is composed of these vectors as columns)*

$$\mathbf{M}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) (i)

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Note : It is important to get this matrix multiplication in the right order. The matrices run from right to left in order of occurence.)

(ii) Rotation of 180 degrees about the z-axis

(Note : Analogous to before, the 1 on the diagonal on the third row indicates rotation around the z-axis. The left upper four values indicate an angle of 180 degrees)

Question 4

(a) Insert P into each equation in turn

$$1+2-2=1$$

$$1+3+2=6$$

(b) Find the vector product of (1,2,-1) and (1,3,1)

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$=(2-(-3))\mathbf{i}-(1-(-1))\mathbf{j}+(3-2)\mathbf{k}$$

$$=5\mathbf{i}-2\mathbf{j}+\mathbf{k}$$

Since (1,1,2) lies on l, required equation is

$$\frac{x-1}{5}=\frac{y-1}{-2}=\frac{z-2}{1}$$

Alternatively, you could eliminate z from both equations

$$2x+5y=7 \Rightarrow y=\frac{7-2x}{5}$$

and eliminate x from both equations

$$y+2z=5 \Rightarrow y=5-2z$$

Combining these two new equations

$$y=5-2z=\frac{7-2x}{5}$$

$$y=-2\left(z-\frac{5}{2}\right)=\frac{7-2x}{5}$$

$$\frac{y}{-2}=\frac{z-\frac{5}{2}}{1}=\frac{x-\frac{7}{2}}{5}$$

(c)

$$\cos\theta=\frac{(5,-2,1).(0,1,0)}{\sqrt{1}\times\sqrt{5^2+(-2)^2+1}}$$

$$=\frac{-2}{\sqrt{29}}$$

and

$$\theta=68.6°$$

Question 5

(a)

$$\begin{bmatrix} 3 & -1 & p \\ 0 & -5 & p \end{bmatrix}\begin{bmatrix} p & -1 \\ -2 & 0 \\ -3 & 3 \end{bmatrix}=\begin{bmatrix} 2 & -3+3p \\ 10-3p & 3p \end{bmatrix}$$

(b) (i)

$$|\mathbf{AB}| = 6\text{p} - (10-3\text{p})(3\text{p}-3)=0$$

$$=6p-(30p-30-9p^2-9p)=0$$

$$=9p^2-33p+30=0$$

$$=3(3p^2-11p+10)=0$$

$$=3(3p-5)(p-2)=0$$

Giving

$$p=2 \;\; or \;\; p=\frac{5}{3}$$

(ii) For p = 2

$$\mathbf{AB}=\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \Rightarrow \mathbf{B^T A^T}=(\mathbf{AB})^{\mathbf{T}}=\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

For $p=\frac{5}{3}$

$$\mathbf{AB}=\begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \Rightarrow \mathbf{B^T A^T}=\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix}$$

(iii) the determinant of **AB** is zero.

(Note : This is clearly seen in that method of calculation of the inverse whereby the inverse requires a rational term with the determinant in the denominator)

Question 6

(a) In order to show that 1 is an eigenvalue, the characteristic equation

$$|\mathbf{M}-\lambda\mathbf{I}|=0$$

must be satisfied. So for $\lambda=1$

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix}=3(2-2)=0$$

So 1 **is** an eigenvalue. To find a corresponding vector

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So

$$4x=x \Rightarrow 3x=0 \Rightarrow x=0$$

$$2y+2z=y \Rightarrow y+2z=0 \Rightarrow y=-2z$$

Giving an eigebvector

$$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

(b) (i)

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left[(2-\lambda)(3-\lambda)-2\right]$$
$$= (4-\lambda)((6+\lambda^2-5\lambda-2)$$
$$= (4-\lambda)(\lambda^2-5\lambda+4)$$

$$(4-\lambda)(\lambda-4)(\lambda-1)$$

so the one other distinct eigenvalue is 4

(ii) *(Note : Using the 'short cut' method in contrast to before)*

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2y+2z \\ y-z \end{bmatrix}$$

$$y = z, \quad x = \ any \ \ value$$

So an appropriate eigenvector is

$$\begin{bmatrix} p \\ q \\ q \end{bmatrix}$$

(c) (i) This would correspond to $\lambda = 1$, e.g. x=0, y=-2z

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix}$$

(ii) This would correspond to $\lambda = 1$, e.g. x=0, y=-2z

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = 4\begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Mechanics 1 - MAM1/W , 12 January 2004

Question 1

(a)

$$\mathbf{F} = \begin{bmatrix} 6 \\ -2.5 \end{bmatrix} \text{N}$$

(b)

$$|\mathbf{F}| = \sqrt{6^2 + (-2.5)^2} = 6.5\text{N}$$

Question 2

(a) diagram, showing the reaction R perpendicular to the slide, the friction F parallel to the slide (upwards, of course) and the weight.

(b) Magnitude of Normal Reaction

$$R = 35g\cos 25 = 311N$$

(c) The friction is given by

$$F = 35g\sin 25 = 145N$$

(Note : The friction will be equal to the component of weight down the slide because Mathew travels at constant speed)

So the coefficient of friction

$$\mu = \frac{F}{R} = \frac{145}{311} = 0.466$$

Question 3

(a) Using

$$v^2 = u^2 + 2as$$

$$v^2 = 5^2 - 2 \times (-1.8) \times 2.5$$

$$v^2 = 16$$

Therefore

$$v = \pm 4ms^{-1}$$

(b) Using

$$t = \frac{v-u}{a}$$

$$= \frac{-4-4}{-1.8}$$

$$= 4.44s$$

Question 4

(a) (i) Retardation

$$= \frac{v-u}{t} = \frac{-9}{6} = -1.5ms^{-2}$$

(ii)

$$s = ut + \frac{1}{2}at^2$$

$$= 9 \times 6 - \frac{1}{2} \times 1.5 \times 36$$

$$= 54 - 27 = 27m$$

(b)

$$v = 9 - \frac{t^2}{4}$$

$$\text{Distance } = \int_0^6 \left(9 - \frac{t^2}{4}\right) dt$$

$$= \left[9t - \frac{1}{12}t^3\right]_0^6$$

$$= \left(54 - \frac{1}{12} \times 6^3\right) - (0)$$

$$= 36m$$

(c) The times are identical, so the average velocity is greatest in (b) where the distance is greatest.

Question 5

(a)

$$\text{Velocity : } \mathbf{v} = (4\text{t}^3 - 4\text{t})\mathbf{i} + (12\text{t}^2 - 4\text{t}^3)\mathbf{j}$$

(b)

$$\text{Momentum} = m\mathbf{v} = \left(\text{t}^3 - \text{t}\right)\mathbf{i} + \left(3\text{t}^2 - \text{t}^3\right)\mathbf{j}$$

(c)

$$\mathbf{F} = \frac{\text{d}}{\text{dt}}(\text{m}\mathbf{v}) = (3\text{t}^2 - 1)\mathbf{i} + (6\text{t} - 3\text{t}^2)\mathbf{j}$$

Alternatively (but longer)

$$\textit{Acceleration } (\mathbf{a}) = (12\text{t}^2 - 4)\mathbf{i} + (24\text{t} - 12\text{t}^2)\mathbf{j}$$

so

$$\mathbf{F} = \text{m}\mathbf{a} = 0.25\mathbf{a} = (3\text{t}^2 - 1)\mathbf{i} + (6\text{t} - 3\text{t}^2)\mathbf{j}$$

(d) **F** acts in direction of **j** when

$$3t^2 - 1 = 0 \Rightarrow t^2 = \frac{1}{3} \Rightarrow t = \pm\frac{1}{3}s$$

(Note : question asks for **exact** *value, so fraction must be present in answer, i.e. not as a decimal)*

Question 6

(a)

$$T = mg = 0.4 \times 9.8 = 3.92N$$

(b) (i) Equation of motion for both masses

$$\text{A:} \quad 0.6g - T = 0.6a$$

$$\text{B:} \quad T - 0.4g = 0.4a$$

Adding

$$0.2g = a$$

thus

$$a = 1.96ms^{-2}$$

(ii) Using

$$v = u + at$$

$$v = 1.96 \times 1.5 \Rightarrow 2.94ms^{-1}$$

(c) After the clay has dropped off, bucket continues at constant velocity of $2.94ms^{-1}$. Its distance at time t is given by

$$2.94t$$

For the clay, its distance is given by

$$s = ut + \frac{1}{2}gt^2 = 2.94t + 4.9t^2$$

So vertical distance between the two objects is given by

$$s = 2.94t + 4.9t^2 - 2.94t = 4.9t^2$$

Question 7

(a) For horizontal component of motion

$$x = u_x t = 7t \qquad (7)$$

For downwards component of motion

$$y = u_y t + \frac{1}{2}gt^2$$

$$y = \frac{1}{2} \times 9.8t^2$$

$$y = 4.9t^2 \qquad (8)$$

So required coordinates are

$$\left(7t, 4.9t^2\right)$$

(b) Using equations from (i) and substituting (7) into (8)

$$y = 4.9\left(\frac{x}{7}\right)^2$$

$$y = \frac{4.9x^2}{49}$$

$$y = \frac{x^2}{10}$$

(c)

$$y = \frac{x^2}{10}$$

$$x^2 = 10y$$

$$\Rightarrow x = \sqrt{10y}$$

Therefore

$$x = \sqrt{81} = 9m$$

(d) For vertical component

$$v^2 = u^2 + 2gs$$

$$v^2 = 2 \times 9.8 \times 8.1 = 158.76$$

Therefore

$$v = 12.6ms^{-1}$$

Since velocity in horizontal direction is 7, its speed is then given by

$$\sqrt{12.6^2 + 7^2}$$

$$= \sqrt{207.76} = 14.4ms^{-1}$$

Mechanics 2 - MAM2/W, 21 January 2004

Question 1

(a) Impulse is given by the area under the graph

Area up to 5 secs

$$= 2000 \times 5 = 10,000Ns$$

Area from 5s to 10s

$$= \frac{1}{2} \times 5 \times 2000 = 5000Ns$$

So total area, i.e. impulse for the 10 second period

$$= 10,000 + 5,000 = 15,000Ns$$

(b) Change in momentum = Impulse

$$1000v - 0 = 15,000$$

$$v = \frac{15,000}{1000} = 15ms^{-1}$$

Question 2

(a) diagram, with the normal reaction N perpendicular to the turntable, the friction F acting parallel to the turntable (towards the center) and the weight 0.01g.

(b) F is given by

$$F = mr\omega^2$$

$$= 0.01 \times 0.4 \times \omega^2$$

$$= 0.004\omega^2$$

(c) If the coin is on point of slipping

$$\mu = \frac{F}{N}$$

$$\mu = \frac{0.004\omega^2}{N}$$

$$\omega^2 = \frac{\mu N}{0.004}$$

$$\omega = \sqrt{\frac{\mu N}{0.004}}$$

so

$$\omega = \sqrt{\frac{0.8 \times 0.01 \times 9.8}{0.004}} = 4.43\ rads/sec$$

Question 3

(a) Elastic potential energy

$$= \frac{1}{2}kx^2 = \frac{1}{2} \times 50 \times (0.03)^2 = 0.0225J$$

(b) Loss in elastic energy = gain in kinetic energy + gain in gravitational PE

$$0.0225 = \frac{1}{2}mv^2 + mgh$$

$$v^2 = \frac{2(0.0225 - mgh)}{m}$$

$$= \frac{2(0.0225 - 0.02)}{0.02 \times 9.8 \times 0.03}$$

$$= 1.662$$

therefore

$$v = 1.29ms^{-1}$$

(c)

$$mgh = 0.0225$$

$$h = \frac{0.0255}{mg} = \frac{0.0255}{0.02 \times 9.8} = 0.1147m \Rightarrow 11.5cm$$

Alternatively (but longer) could equate PE and KE after it has travelled 3 cms (i.e. when elastic force ceases to have effect)

gain in potential energy = loss in kinetic energy

$$mgh = \frac{1}{2}mv^2$$

so

$$h = \frac{1}{2}\frac{v^2}{g}$$

$$h = \frac{1}{2}\left(\frac{1.29^2}{9.8}\right) = 0.0849$$

Adding the 3 cms gives 0.1149, i.e. 11.5 cms to 3 sig. figs.

Could also arrive at the correct answer by using the equation $v^2 = u^2 + 2as$, but this is probably not recommended

Question 4

(a) 5cm

(b) Taking moments about SP, using

$$M_t\bar{x} = \sum_i m_i x_i$$

$$M\bar{x} = m_{plate} \times 3 + m_R \times 6 + m_Q \times 6$$
$$M\bar{x} = 5 + 6m + 6m = 12m + 5$$

Therefore, since $M_t = 2m + 1$.

$$\bar{x} = \frac{12m+3}{2m+1}$$

(c) Since the vertical line through P must pass through the center of mass at a point 5 cms from PQ, we can construct a triangle with one side of 5cm, and angles of 45, 90 and thus another 45. Therefore, by inspection, the center of mass is 5cm from PS. Thus

$$\frac{12m+3}{2m+1} = 5 \Rightarrow m = 1$$

Question 5

(a) (i) Loss in potential energy = gain in kinetic energy

$$mgh = \frac{1}{2}mv^2$$
$$v = \sqrt{2gh}$$
$$= \sqrt{2 \times 9.8 \times 5\cos 30}$$
$$= \sqrt{2 \times 9.8 \times \frac{5\sqrt{3}}{2}}$$
$$= 9.21ms^{-1}$$

(ii) Resolve forces towards O and equate to the formula for central force

$$T - mg\cos 30 = \frac{mv^2}{r}$$
$$T = 70 \times 9.8 \times \frac{\sqrt{3}}{2} + \frac{70 \times 9.21^2}{5}$$
$$= 1781.6N \Rightarrow 1780N \text{ to 3 sig figs}$$

(b) Using, in a vertical direction

$$v^2 = u^2 - 2gd$$
$$0 = (9.21\sin 30)^2 - 2 \times 9.8 \times d$$
$$d = \frac{(9.21\sin 30)^2}{2 \times 9.8} = 1.08m$$

i.e. $\approx 1.00m$

(c) Modelling the man as a particle could possibly be imperfect

Given the length of rope being 5m, the height of the man ($\approx 2m$, probably) could be a serious negative factor, depending on his attitude while swinging on the rope.

Air resistance might be an extra factor, again depending on attitude

Question 6

(a) Restitution

$$v_B - v_A = eu \quad (9)$$

Conservation of Momentum (given that both A and B are of the same mass

$$u = v_A + v_B \quad (10)$$

Adding (9) and (10)

$$u = v_B - eu + v_B$$
$$2v_B = u + eu$$
$$v_B = \frac{1}{2}u(1+e) \quad (11)$$

and from (9) *(for example)*

$$v_A = \frac{1}{2}u(1+e) - eu$$
$$v_A = \frac{1}{2}u(1-e) \quad (12)$$

(b) After it hits the wall

$$v_B = \frac{2}{3} \times \frac{1}{2}u(1+e) = \frac{1}{3}u(1+e)$$

(c) (i) Equating v_A to v_B

$$\frac{1}{2}u(1-e) = \frac{1}{3}u(1+e)$$
$$\frac{1}{2} - \frac{1}{2}e = \frac{1}{3} + \frac{1}{3}e$$
$$\frac{1}{6} = \frac{5}{6}e$$
$$e = \frac{1}{5}$$

(ii) Therefore speed of both spheres (from (12)

$$= \frac{1}{2}u\left(1 - \frac{1}{5}\right) = \frac{2}{5}u$$

(d) Speed of B

$$v_B = \frac{1}{2}u(1+e) = \frac{3}{5}u$$

B reached wall after

$$\frac{5d}{3u}s$$

In this time A has travelled

$$\frac{2}{5}u \times \frac{5d}{3u}$$
$$= \frac{2d}{3}$$

Second collision now occurs at halfway between A and B

$$\frac{d - \frac{2d}{3}}{2}$$
$$= \frac{d}{6}$$

Mechanics 3 - MAM3 , 27 January 2004

Question 1

(a) Distance dropped = the circumference of the flywheel, i.e.

$$2\pi r = 2\pi \times 0.2 = 1.26m$$

(0.4π *would probably be OK - it's definitely better to use* 0.4π *as the distance in the subsequent calculations in (b), because this is the* **exact** *distance)*

(b) Loss in PE of mass = Gain in KE of mass + Gain in KE of flywheel

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$5 \times 9.8 \times 0.4\pi = \frac{1}{2}.5v^2 + \frac{1}{2}.10\omega^2$$

$$5 \times 9.8 \times 0.4\pi = \frac{1}{2}.5(0.2\omega)^2 + \frac{1}{2}.10\omega^2$$

$$19.6\pi = 0.1\omega^2 + 5\omega^2$$

$$5.1\omega^2 = 19.6\pi$$

$$\omega = \sqrt{\frac{19.6\pi}{5.1}} = 3.47rad/sec$$

Question 2

(a) Diagram showing all forces, with R normal reaction at B and F the friction at this point

Resolving vertically

$$R = 4W + W = 5W$$

Since the ladder is on point of moving

$$F = \mu R = \frac{1}{4} \times 5W = \frac{5W}{4}$$

(b) Taking moments about A

$$4Wa\cos\theta + W.2a\cos\theta + P.3a\sin\theta + F.4a\sin\theta = R.4a\cos\theta$$

$$6W\cos\theta + 3P\sin\theta + 4F\sin\theta = 4R\cos\theta$$

$$6W\frac{5}{13} + 3P\frac{12}{13} + 4.\frac{5W}{4}.\frac{12}{13} = 4(5W)\frac{5}{13}$$

$$30W + 36P + 60W = 100W$$

$$36P = 10W$$

$$P = \frac{10W}{36} = \frac{5W}{18}$$

Question 3

(a) Area of strip

$$2\pi r\delta r$$

Mass of strip

$$2\pi\rho r\delta r$$

Moment of inertia of strip

$$2\pi\rho r^3\delta r$$

To find required M of I of disk (I), integrate across all strips from a to 2a

$$I = \int_a^{2a} 2\pi\rho r^3\, dr$$

$$= 2\pi\rho\left[\frac{r^4}{4}\right]_a^{2a}$$

$$= 2\pi\rho\left\{(4a^4) - \left(\frac{a^4}{4}\right)\right\}$$

$$= \frac{30\pi\rho a^4}{4} = \frac{15\pi\rho a^4}{2}$$

now, from above, total mass M is given by

$$M = \int_a^{2a} 2\pi\rho r\, dr$$

$$= 2\pi\rho\left[\frac{r^2}{2}\right]_a^{2a}$$

$$= 2\pi\rho\left\{\frac{4a^2}{2} - \frac{a^2}{2}\right\} = 3\pi\rho a^2$$

so I becomes

$$I = \frac{5Ma^2}{2}$$

(b) Using the Perpendicular Axis Theorem

$$I_z = I_x + I_y$$

by symmetry

$$I_z = 2I_x = 2I_y$$

therefore, I about a diameter (i.e. either I_x *or* I_y) is given by

$$I_D = \frac{I_z}{2} = \frac{5Ma^2}{4}$$

Question 4

(a) Resolving vertically

$$5 + 4 + 5.\frac{3}{5} = 12$$

Resolving horizontally

$$-6 + 7 + 5.\frac{4}{5} = 5$$

Therefore resultant

$$= \sqrt{12^2 + 5^2} = \sqrt{169} = 13$$

(b) (i) Consider the horizontal component F_x cutting the y axis clockwise at a point with y-coordinate y_0. Then taking moments about O

$$F_x y_0 = -4 \times 4 - 6 \times 3 + 19$$
$$5y_0 = -16 - 18 + 19$$
$$y_0 = \frac{-15}{5} = -3$$

i.e the line of action of the resultant cuts the y-axis at (0,-3)

(ii) From (a), gradient of line $=\frac{12}{5}$, so equation of line is

$$y - y_0 = \frac{12}{5}(x - x_0)$$

using the coordinates from (i)

$$y + 3 = \frac{12}{5}(x - 0)$$
$$y = \frac{12}{5}x - 3$$

Question 5

diagram with Friction F and Normal Reaction R

(a) If the cube is moving

$$P > F \quad where \quad F = \mu R$$

Resolving vertically

$$W = R$$

so

$$P > F \Rightarrow P > \mu R \Rightarrow P > \mu W$$

(b) (i) diagram, showing R acting at A, plus F, P and W

(ii) Taking moments about A

$$P \times 3a \geq W \times 2a$$
$$P \geq \frac{2W}{3}$$

(Using the $\geq$ sign because the question does not state whether the system is moving or in equilibrium)

(c) If $\mu = 0.6$, block will slide if

$$P > 0.6W$$

Since P needs to be $\frac{2}{3}W$ before it topples, it will slide first

Question 6

(a) For a rod, Moment of Inertia about an end

$$= \frac{4}{3}Ml^2$$
$$= \frac{4}{3} \times 3m \times l^2$$
$$= 4ml^2$$

(b) (i) Collision elastic, so coefficient of restitution = 1

separation speed = approach speed

$$l\omega - v = u \Rightarrow l\omega = u + v$$

(ii) Angular momentum after = Angular momentum before

$$I\omega + 5mvl = 0 + 5mul$$
$$4ml^2.\omega + 5mvl = 5mul$$
$$5mvl = 5mul - 4ml^2\omega$$
$$5v = 5u - 4l\omega$$
$$5v = 5u - 4.(u + v)$$
$$9v = u$$
$$v = \frac{u}{9}$$

(iii) Same direction as before collision

(c) From b(i)

$$l\omega = u + \frac{u}{9}$$
$$= \frac{10}{9}u$$
$$\omega = \frac{10u}{9l}$$

Mechanics 4 - MAM4/W, 29 January 2004

Question 1

(a) From F=ma

Weight - Air Resistance = ma

$$mg - 0.1mv = ma$$

$$g - 0.1v = a$$

i.e.

$$\frac{dv}{dt} = g - 0.1v$$

(b)

$$\frac{dv}{g - 0.1v} = dt$$

$$-10\int \frac{-0.1dv}{g - 0.1v} = \int dt$$

$$-10\left(\ln(g - 0.1v)\right) = t + c$$

When t=0, v= 0

$$c = -10(\ln g)$$

so

$$-10\left(\ln(g - 0.1v)\right) = t - 10(\ln g)$$

$$10 \times \ln\left(\frac{g}{g - 0.1v}\right) = t$$

$$\frac{g}{g - 0.1v} = e^{\frac{t}{10}}$$

$$\frac{g}{e^{\frac{t}{10}}} = g - 0.1v$$

$$v = 10(g - ge^{-\frac{t}{10}})$$

$$v = 10g(1 - e^{-\frac{t}{10}})$$

(c) In the above expression, as $t \to \infty$

$$v \to 10g = 98ms^{-1}$$

Question 2

(a) Force on meteorite

$$F = m_m\mathbf{a} = -\frac{\mathrm{Gm_Em_m}}{\mathrm{r}^2}$$

$$\mathbf{a} = -\frac{\mathrm{Gm_E}}{\mathrm{r}^2}$$

$$= -\frac{6.7 \times 10^{-11} \times 6 \times 10^{24}}{x^2}$$

$$= -\frac{4.02 \times 10^{14}}{x^2}ms^{-2}$$

which has a magnitude of

$$\approx \frac{4 \times 10^{14}}{x^2}$$

(b)

$$ma = v\frac{dv}{dx} = -\frac{4.02 \times 10^{14}}{x^2}$$

$$vdv = -\frac{4.02 \times 10^{14}}{x^2}dx$$

$$\int vdv = -4.02 \times 10^{14}\int \frac{1}{x^2}dx$$

$$\left[\frac{v^2}{2}\right]_{50}^{v} = -4.02 \times 10^{14}\left[\frac{x^{-1}}{-1}\right]_{1.44\times10^7}^{6.4\times10^6}$$

(Note: 1.44×10^7 *is the distance to the center of the Earth)*

$$\left(\frac{v^2}{2} - \frac{50^2}{2}\right) = 4.02 \times 10^{14}\left(\frac{1}{6.4 \times 10^6} - \frac{1}{1.44 \times 10^7}\right)$$

$$v = \sqrt{8.04 \times 10^{14}\left(\frac{1}{6.4 \times 10^6} - \frac{1}{1.44 \times 10^7}\right) + 50^2}$$

$$v = 8.33 \times 10^3ms^{-1}$$

Question 3

(a) Resolve perpendicular to string and equate to 'ma'

$$mg\sin\theta + 2mv = -mr\ddot{\theta}$$

$$g\sin\theta + 2(r\dot{\theta}) = -r\ddot{\theta}$$

$$9.8\sin\theta + 2(0.98)\dot{\theta} = -0.98\ddot{\theta}$$

$$\ddot{\theta} + 2\dot{\theta} + 10\sin\theta = 0$$

Using the small angle approximation, i.e. $\sin\theta \approx \theta$

$$\ddot{\theta} + 2\dot{\theta} + 10\theta = 0$$

(b) The auxiliary eqn is

$$m^2 + 2m + 10 = 0$$

with roots

$$m = \frac{-2 \pm \sqrt{4 - 40}}{2}$$

$$= -1 \pm i3$$

giving the General Solution

$$\theta = e^{-t}\{A\cos 3t + B\sin 3t\}$$

(c) When t = 0, $\theta = \frac{\pi}{20}$

$$\frac{\pi}{20} = A$$

Differentiating

$$\dot{\theta} = e^{-t}(-3A\sin 3t + 3B\cos 3t) - e^{-t}\{A\cos 3t + B\sin 3t\}$$

When t=0, $\dot{\theta} = 0$

$$0 = 3B - A$$

$$B = \frac{1}{3}A = \frac{1}{3} \times \frac{\pi}{20}$$

giving the particular solution

$$\theta = \frac{\pi}{20}e^{-t}\{\cos 3t + \frac{1}{3}\sin 3t\}$$

(d)

$$\theta = \frac{\pi}{20}e^{-t}\{\cos 3t + \frac{1}{3}\sin 3t\}$$

and

$$\dot{\theta} = \frac{\pi}{20}e^{-t}(-3\sin 3t + \cos 3t) - \frac{\pi}{20}e^{-t}\left(\cos 3t + \frac{1}{3}\sin 3t\right)$$

$$= \frac{\pi}{20}e^{-t}\left(-3\sin 3t + \cos 3t - \cos 3t - \frac{1}{3}\sin 3t\right)$$

$$= -\frac{\pi}{20}e^{-t}\left(\frac{10}{3}\sin 3t\right)$$

When $\dot{\theta} = 0$

$$\frac{10}{3}\sin 3t = 0$$

$$\sin 3t = 0$$

for first time at rest after motion starts

$$3t = \pi$$

$$t = 1.05$$

Question 4

(a) Using impulse/momentum principle

$$(m + \delta m)(v + \delta v) - mv = -F\delta t$$

$$m\delta v + v\delta m = -F\delta t$$

ignoring the small term $\delta m \delta v$

when $\delta t \to 0$

$$m\frac{dv}{dt} + v\frac{dm}{dt} = -F$$

inserting numbers

$$(40 + 0.05t)\frac{dv}{dt} + 0.05v = -5$$

(b) re-arranging

$$\int \frac{dv}{(5 + 0.05v)} = -\int \frac{dt}{(40 + 0.05t)}$$

$$20\ln(5 + 0.05v) = -20\ln(40 + 0.05t) + C$$

$$\ln(5 + 0.05v) + \ln(40 + 0.05t) = D$$

when $t = 0, v = 8$

$$D = \ln(5.4 \times 40) = \ln 216$$

so

$$\ln(5 + 0.05v) + \ln(40 + 0.05t) = \ln 216$$

$$\ln(200 + 0.25t + 2v + (0.05)^2vt) = \ln 216$$

$$200 + 0.25t + 2v + (0.05)^2vt = 216$$

$$v(2 + (0.05)^2t) = 16 - 0.25t$$

$$v = \frac{16 - 0.25t}{2 + (0.05)^2t}$$

or

$$v = \frac{64 - t}{8 + 0.01t}$$

Question 5

(a) If there is no transverse force, angular momentum is conserved

$$\frac{1}{r}\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

for a fixed mass

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$\Rightarrow r^2\dot{\theta} = h$$

where h is a constant

(b)

$$r = \frac{1}{1 + a\cos\theta} = (1 + a\cos\theta)^{-1}$$

$$\dot{r} = -(1 + a\cos\theta)^{-2}(-a\sin\theta)\dot{\theta}$$

$$= a\left(\frac{1}{(1 + a\cos\theta)^2}\dot{\theta}\right)\sin\theta$$

$$= ah\sin\theta$$

(c) The radial force is given by

$$F_r = -m\left(\ddot{r} - r\dot{\theta}^2\right)$$

now

$$\ddot{r} = ah(\cos\theta)\,\dot{\theta}$$

so

$$F_r = -m\left(h\left(\frac{1}{r} - 1\right)\frac{h}{r^2} - r\left(\frac{h}{r^2}\right)^2\right)$$

$$= -m\left(\frac{h^2}{r^3} - \frac{h^2}{r^2} - \frac{h^2}{r^3}\right)$$

$$F = m\frac{h^2}{r^2}$$

(This derivation made use of

$$a\cos\theta = \frac{1}{r} - 1$$

from equation for r given in the question, and

$$\dot{\theta} = \frac{h}{r^2}$$

from the equation derived in (a))