AQA Board. **GCE Maths (Specification A).**

Model Answers

January 2003

Note comment on paper about stating final answers to 3 significant figures, in general

Pure 1 - MAP1, 15 January 2003

Question 1

(a)

$$\frac{9}{10} = \frac{8.1}{9} = \frac{7.29}{8.1} = 0.9$$

(b)

$$u_n = ar^{n-1}$$
$$= 10\left(0.9^{n-1}\right)$$

(c)

$$S_n = a\frac{1-r^n}{1-r}$$
$$= 10.\frac{1-0.9^{25}}{1-0.9}$$
$$= 10(9.282)$$
$$= 92.8 \text{ to 3 sig figs}$$

(d)

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{10}{0.1} = 100$$

Question 2

(a) At P

$$x^3 = x + 1$$
$$x^3 - x - 1 = 0$$

(b) (i) When x = 1.2

$$1.2^3 - 1.2 - 1 = -0.472$$

When x=1.4

$$1.4^3 - 1.4 - 1 = 0.344$$

i.e. the expression changes sign between these two points, therefore P must lie within this interval

(ii) when x = 1.3

$$1.3^3 - 1.3 - 1 = -0.103$$

when x = 1.35

$$1.35^3 - 1.35 - 1 = 0.110375$$

which is too high

$$1.3 < x < 1.35$$

(iii) so to 1 d.p

$$x = 1.3$$

Question 3

(a)

$$\frac{dy}{dx} = \frac{1}{x} - 3$$
$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

(b) Set

$$\frac{1}{x} - 3 = 0$$
$$x = \frac{1}{3}$$

(c) at $x = \frac{1}{3}$

$$\frac{d^2y}{dx^2} = -\frac{1}{\left(\frac{1}{3}\right)^2}$$
$$= -9$$

so stationary point is a maximum

Question 4

(a) (i)

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos^2\theta = 1 - \left(\frac{5}{13}\right)^2$$
$$\cos^2\theta = \frac{144}{169} \Rightarrow \cos\theta = \frac{12}{13}$$

Could also use Pythagoras. If the hypotenuse is 13 and the opposite side is 5, then the adjacent side equals 12.

(ii)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{5}{12}$$

(b)

$$\theta = 0.395 \text{ to 3 d.p.}$$

(c) (i)

$$5 \approx 0.395r$$
$$r \approx \frac{5}{0.395} \approx 12.658 \approx 12.7$$

(ii) Area

$$= \frac{\theta r^2}{2} = 0.395 \times \frac{12.7^2}{2}$$
$$= 32cm^2$$

Question 5

(a) (i)

$$(0, 5)$$

(ii)

$$0 < f(x)$$

(iii)

$$f(\ln 6) = 5e^{-\ln 6} = 5e^{\ln 6^{-1}}$$

$$= \frac{5}{6}$$

(b) (i)

$$gf(x) = 5e^{-x} + 10$$

$$= 5(e^{-2x} + 2)$$

(ii)

$$gf(x) > 10$$

(iii) same as given in question but translated upwards by 10 units

(iv)

$$5(e^{-x} + 2) = 11$$

$$e^{-x} + 2 = 2.2$$

$$e^{-x} = 0.2$$

$$-x = \ln 0.2$$

$$x = -\ln 0.2$$

$$x = \ln 5$$

(c) (i)

$$15°C$$

(ii)

$$11 = 5(e^{-t} + 2)$$

from (b)(iv)

$$t = \ln 5$$

$$t = 1.6\ min \text{ to one d.p.}$$

Question 6

(a) (i)

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

(ii) At x=4

$$f'(x) = \frac{1}{4}$$

which is the gradient at that point

(b) (i)

$$\int f(x)dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2x + C$$

$$= \frac{2x^{\frac{3}{2}}}{3} + 2x + C$$

(ii)

$$\left[\frac{2x^{\frac{3}{2}}}{3} + 2x\right]_0^4$$

$$= \left(\frac{16}{3} + 8\right) - (0)$$

$$= \frac{40}{3}$$

(c) Let

$$y = x^{\frac{1}{2}} + 2$$

$$x^{\frac{1}{2}} = y - 2$$

$$x = (y-2)^2$$

which is the same as

$$f^{-1}(x) = (x-2)^2$$

(d) (i)

$$y = x$$

(ii) Area under $f^{-1}(x)$

$$= \int_2^4 (x-2)^2$$

$$= \left[\frac{(x-2)^3}{3}\right]_2^4$$

$$= \frac{8}{3}$$

so area of A (using result from (b)(ii)

$$= \frac{40}{3} - \frac{8}{3}$$

$$= \frac{32}{3}$$

Pure 2 - MAP2, 20 January 2003

Question 1

$$(-3)^3 + (-3)^2p - 3 + 54 = -3$$
$$-27 + 9p - 3 + 54 = -3$$
$$9p = -27$$
$$p = -3$$

Question 2

(a)

$$\int_0^6 \frac{1}{2+u}du$$
$$= [\ln(2+u)]_0^6$$
$$= \ln 8 - \ln 2$$
$$= \ln 4$$

(b) Substituting

$$x = u^2 \quad and \quad \frac{dx}{du} = 2u$$

integral becomes

$$\int_0^6 \frac{1}{u(2+u)}2udu$$
$$= \int_0^6 \frac{2}{2+u}du$$
$$= 2\left[\ln(2+u)\right]_0^6$$
$$= 2\left(\ln 8 - \ln 2\right)$$
$$= 2\ln 4$$
$$= \ln 16$$

Question 3

(a) Substituting coordinates of A into LHS of equation

$$\left(\frac{3}{5} - 3\right)^2 + \left(\frac{4}{5} - 4\right)^2$$
$$= \frac{9}{25} - \frac{18}{5} + 9 + \frac{16}{25} - \frac{32}{5} + 16$$
$$= 1 - \frac{50}{5} + 9 + 16$$
$$= 16$$

(b) Circle center (3,4), radius 4

(c) Differentiating equation for circle

$$2(x-3) + 2(y-4)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{(x-3)}{(y-4)}$$

At A, the gradient

$$= -\frac{-\frac{12}{5}}{-\frac{16}{5}} = -\frac{3}{4}$$

so gradient of normal

$$= \frac{4}{3}$$

equation of normal at A

$$y - \frac{4}{5} = \frac{4}{3}\left(x - \frac{3}{5}\right)$$
$$y - \frac{4}{5} = \frac{4}{3}x - \frac{4}{5}$$
$$y = \frac{4}{3}x$$

which passess thru the origin

(d)

$$y - \frac{4}{5} = -\frac{3}{4}\left(x - \frac{3}{5}\right)$$
$$y - \frac{4}{5} = -\frac{3}{4}x + \frac{9}{20}$$
$$\frac{3}{4}x + y = \frac{25}{20}$$
$$3x + 4y = 5$$

Question 4

(a) (i)

$$L = 2\sin\theta + 4\cos\theta$$

(ii) let

$$2\sin\theta + 4\cos\theta = R\sin(\theta + \alpha)$$

so

$$2\sin\theta + 4\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

therefore

$$R\cos\alpha = 2$$

$$R\sin\alpha = 4$$

so

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 2^2 + 4^2$$

$$R^2 = 20 \Rightarrow R = \sqrt{20}$$

(*Note : this used*

$$\cos^{\alpha} + \sin^2\alpha = 1$$

)

and

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = 2$$

$$\alpha = \tan^{-1} 2 = 1.107$$

so required form is

$$\sqrt{20}\sin(\theta + 1.107)$$

(b) (i)

$$L_{max} = \sqrt{20}$$

(ii) at maximum

$$\sin(\theta + 1.107) = 1$$

$$\theta + 1.107 = \frac{\pi}{2}$$

$$\theta = 0.46\ rad \text{ to 2 d.p.}$$

Question 5

(a)

$$A\left(1, \frac{\pi}{2}\right)$$

$$B\left(-1, -\frac{\pi}{2}\right)$$

(b) With 5 strips

x	**y**
0.1	0.1002
0.3	0.3047
0.5	0.5236
0.7	0.7754
0.9	1.1198

$$I \approx \frac{1}{5}(0.1002 + 0.3047 + 0.5236 + 0.7754 + 1.1198)$$

$$\approx 0.565 \text{ to 3 d.p.}$$

Question 6

(a) Let $x = -4$ which is outside of the student's solution

$$\frac{2x+3}{x} = \frac{-5}{-4} = 1.25$$

which is > 1 and so $x = -4$ is allowed by the inequality

(b) S(he) has multiplied thru by x, whereas you cannot multipy thru by a value without knowing whether it is positive or negative - if it is negative the inequality sign needs to be reversed.

(c)

$$\frac{2x+3}{x} > 1$$

$$\frac{x^2(2x+3)}{x} > x^2$$

$$2x^2 + 3x > x^2$$

$$x^2 + 3x > 0$$

$$x(x+3) > 0$$

	$x < -3$	$3 < x < 0$	$0 < x$
x	$-$	$-$	$+$
x+3	$-$	$+$	$+$
x(x+3)	$+$	$-$	$+$

So

$$x < -3 \text{ and } x > 0$$

Question 7

(a) (i)

$$\frac{dy}{dx} = -x\sin x + \cos x$$

(ii) At max

$$-x\sin x + \cos x = 0$$

$$x\sin x - \cos x = 0$$

$$x\tan x - 1 = 0$$

$$\tan x = \frac{1}{x}$$

therefore

$$x = \tan^{-1}\frac{1}{x}$$

(iii)

$$x_1 = 0.9$$

$$x_2 = 0.837..$$

$$x_3 = 0.873..$$

$$x_4 = 0.852..$$

$$x_5 = 0.864..$$

$$x_6 = 0.857..$$

$$x_7 = 0.861..$$

$$x_8 = 0.859..$$

$$x_9 = 0.860..$$
$$x_{10} = 0.860..$$
$$x_{11} = 0.860..$$

so

$$x = 0.86 \text{ to } 2 \text{ d.p.}$$

(b)

$$Area = \int_0^{\frac{\pi}{2}} x \cos x dx$$

$$= [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx$$
$$= \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2} + (0 - 1)$$
$$= \frac{\pi}{2} - 1$$

Pure 3 - MAP3, 24 January 2003

Question 1

(a)

$$1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

(b) (i)

$$(4+2x)^{\frac{1}{2}} = \left(4\left[1+\frac{x}{2}\right]\right)^{\frac{1}{2}}$$
$$= 2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}$$
$$= 2\left(1+\frac{1}{4}x - \frac{1}{32}x^2\right)$$
$$= 2 + \frac{1}{2}x - \frac{1}{16}x^2$$

(ii) Need

$$\left|\frac{x}{2}\right| < 1$$
$$|x| < 2$$

Question 2

(a)

$$x = 3\sin t \Rightarrow \frac{dx}{dt} = 3\cos t$$
$$y = \cos t \Rightarrow \frac{dy}{dt} = -\sin t$$

gradient

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\sin t . \frac{1}{3\cos t} = -\frac{1}{3} tant$$

at $t = \frac{\pi}{4}$, the gradient

$$= -\frac{1}{3}\tan\frac{\pi}{4} = -\frac{1}{3}$$

(b) When $t = \frac{\pi}{4}$

$$x = 3\sin\frac{\pi}{4} = \frac{3}{\sqrt{2}}$$
$$y = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

so required equation is

$$y - \frac{1}{\sqrt{2}} = -\frac{1}{3}\left(x - \frac{3}{\sqrt{2}}\right)$$
$$y = -\frac{1}{3}x + \frac{2}{\sqrt{2}}$$
$$y = -\frac{1}{3} + \sqrt{2}$$

Question 3

(a) (i) *You might be thinking of doing the first step of a'Partial fractions' procedure, but it appears here they just want you to show that the LHS=RHS*

$$1 + \frac{16}{x^2-16}$$
$$= \frac{x^2-16}{x^2-16} + \frac{16}{x^2-16}$$
$$= \frac{x^2}{x^2-16}$$

(ii)

$$\frac{16}{x^2-16}$$

$$= \frac{16}{(x-4)(x+4)} = \frac{A}{x-4} + \frac{B}{x+4}$$
$$16 = A(x+4) + B(x-4)$$

Let $x = 4$

$$16 = 8A \Rightarrow A = 2$$

Let $x = -4$

$$16 = -8B$$
$$B = -2$$

giving

$$\frac{2}{x-4} - \frac{2}{x+4}$$

(b) integral becomes

$$\int_5^8 \left(1 + \frac{2}{x-4} - \frac{2}{x+4}\right) dx$$
$$\left[x + 2\left(\ln(x-4)\right) - 2\left(\ln(x+4)\right)\right]_5^8$$
$$(8 + 2\ln 4 - 2\ln 12) - (5 + 2\ln 1 - 2\ln 9)$$
$$3 + \ln 16 - \ln 144 + \ln 81$$
$$3 + \ln\frac{81\times 16}{144}$$
$$3 + \ln 9$$

(which would be acceptable), could also state the answer as

$$3 + 2\ln 3$$

Question 4

$$\frac{dy}{dx} = \sqrt{x^2 - 5}; \qquad dy = \frac{dy}{dx}\ dx$$

x	y	$\frac{dy}{dx}$	dx	dy
3	1	2	0.5	1
3.5	2	2.692	0.5	1.346
4.00	3.346			

Therefore

$$y = 3.35 \text{ to 2 d.p.}$$

Method : The initial values of x and y are given . $\frac{dy}{dx}$ is derived from the formula above (left), dx is a constant 0.5, dy is derived from the formula above (right).

Then loop, x goes up in steps of 0.5, and $y_{n+1} = y_n + dy$

Question 5

(a) (i)

$$f(x) = \cos 2x$$
$$f'(x) = -2\sin 2x$$
$$f''(x) = -4\cos 2x$$

(ii)

$$f(0) = 1; \quad f'(0) = 0; \quad f''(0) = -4$$

MacLaurin series for $\cos 2x$ for small values of x (ie. ignoring terms above second degree) is

$$f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$$
$$= 1 + \frac{x^2}{2}(-4)$$
$$= 1 - 2x^2$$

(b)

$$e^{-2x} \approx 1 + (-2x) + \frac{(-2x)^2}{2}$$
$$\approx 1 - 2x + 2x^2$$

therefore

$$e^{-2x} + \sin x \approx 1 - 2x + 2x^2 + x$$
$$\approx 1 - x + 2x^2$$

(c) An approx. soln can be derived from

$$1 - x + 2x^2 = 1 - 2x^2$$
$$4x^2 - x = 0$$
$$x(4x - 1) = 0$$

which gives

$$x = \frac{1}{4}$$

as the only solution for $x > 0$

Question 6

(a) Equating ratios of vertical/horizontal

$$\frac{x+s}{5.2} = \frac{s}{1.6}$$
$$1.6(x + s) = 5.2s$$
$$1.6x + 1.6s = 5.2s$$
$$3.6s = 1.6x$$
$$s = \frac{1.6x}{3.6} = \frac{4x}{9}$$

(b)

$$\frac{ds}{dx} = \frac{4}{9}$$

now

$$\frac{ds}{dt} = \frac{ds}{dx}.\frac{dx}{dt}$$
$$= \frac{4}{9}.6$$
$$= \frac{8}{3}ms^{-1}$$

Question 7

(a) When $T = -7°$

$$\frac{dx}{dt} = \frac{7}{14000x} = \frac{1}{2000x}$$

(b)

$$\int_2^x 2000x\ dx = \int_{t_0}^t dt$$
$$\left[1000x^2\right]_2^x = [t]_{t_0}^t$$
$$1000x^2 - 4000 = t - t_0$$
$$t = 1000x^2 - 4000 + t_0$$

When x = 3 (and t_0 assigned as zero)

$$t = 1000(3^2) - 4000 = 5000s$$

This is 5000s after 12 noon

$$5000s = \frac{5000}{60} = 83\frac{1}{3} \text{ mins after 12:00}$$
$$= 13:23:20$$

Question 8

(a) (i) Equation of Π is

$$3x - 2y + z = 5$$

Insert coordinates of P into LHS

$$3(2) - 2(-1) + (-3) = 6 + 2 - 3 = 5$$

(ii) Let

$$\begin{aligned} 3+2t &= -1 \\ -2-3t &= 4 \\ 3+4t &= -5 \end{aligned}$$

top line gives

$$t = -2$$

and this value of t satisfies other two lines. Therefore Q does lies on the line l.

(b) To find angle between the normal to Π and the line l

Normal to Π

$$= \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\cos\theta = \frac{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} . \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}}{\sqrt{14}\sqrt{29}}$$

$$\cos\theta = \frac{16}{\sqrt{14}\sqrt{29}}$$

$$= 0.794..$$

therefore

$$\theta = 37.4^\circ$$

and so angle between plane Π itself and line l

$$= 52.6^\circ$$

(c) line from P to line l has the equation

$$\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+2t \\ -1-3t \\ 6+4t \end{bmatrix}$$

Since this line is ⊥ to line l

$$\begin{bmatrix} 1+2t \\ -1-3t \\ 6+4t \end{bmatrix} . \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 0$$

$$2 + 4t + 3 + 9t + 24 + 16t = 0$$

$$29t = -29 \Rightarrow t = -1$$

so required coordinates on line l are

$$(1, 1, -1)$$

Pure 4 - MAP4, 20 January 2003

Question 1

(a) (i)

$$-2$$

(ii)

$$5$$

(b) (i)

$$(\alpha+\beta+\gamma)(\alpha+\beta+\gamma)=4$$
$$\alpha^2+\alpha\beta+\alpha\gamma+\alpha\beta+\beta^2+\beta\gamma+\gamma\alpha+\gamma\beta+\gamma^2=4$$
$$\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)=4$$
$$\alpha^2+\beta^2+\gamma^2+10=4$$
$$\alpha^2+\beta^2+\gamma^2=-6$$

(ii) $\alpha^2+\beta^2+\gamma^2$ would be positive if all roots were real. Real coefficients in the cubic equation mean that complex roots will occur in conjugate pairs.

(c)

$$(-2+3i)^2+(-2-3i)^2+\gamma^2=-6$$
$$4+9i^2-12i+4+9i^2+12i+\gamma^2=-6$$
$$8-18+\gamma^2=-6$$
$$\gamma^2=4$$
$$\gamma=2$$

now

$$(-2+3i)(-2-3i)(2)=-k$$
$$2(4+9)=-k$$
$$k=-26$$

Question 2

(a) (i)

$$z_2=-\sqrt{3}+i$$

(ii) Modulus

$$=\sqrt{(\sqrt{3})^2+1^2}=\sqrt{3+1}=\sqrt{4}=2$$

Argument

$$=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\frac{5\pi}{6}$$

(b) Argand Diagram

(c) (i) line is a bisector between z_1 and z_2 (going thru the origin, as can be deduced by examining the moduli of z_1 and z_2.

(ii) line thru z_1, parallel to z_2

Hint : These lines are calculated in an analogous way to vectors, e.g analogous to $\vec{AB}=\mathbf{b}-\mathbf{a}$

Question 3

(a)

$$\frac{1}{(r-1)(r+1)}=\frac{A}{r-1}+\frac{B}{r+1}$$
$$1=A(r+1)+B(r-1)$$

Let r=1

$$1=2A\Rightarrow A=\frac{1}{2}$$

Let r=-1

$$1=-2B$$
$$B=-\frac{1}{2}$$

so expression in partial fractions is

$$\frac{1}{2(r-1)}-\frac{1}{2(r+1)}$$

(b)

$$\sum_{r=2}^{n}\frac{1}{(r^2-1)}$$
$$=\frac{1}{2}\left(\frac{1}{2(2-1)}-\frac{1}{2(2+1)}\right)$$
$$+\frac{1}{2}\left(\frac{1}{2(3-1)}-\frac{1}{2(3+1)}\right)$$
$$+\frac{1}{2}\left(\frac{1}{2(4-1)}-\frac{1}{2(4+1)}\right)$$

..
..

$$+\frac{1}{2}\left(\frac{1}{2(n-2)}-\frac{1}{2(n)}\right)$$
$$+\frac{1}{2}\left(\frac{1}{2(n-1)}-\frac{1}{2(n+1)}\right)$$

You can see that all terms will cancel out apart from terms at the beginning of first two 'lines' and terms at end of last two 'lines, i.e.

$$\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$$

Question 4

(a) substituting given definition into LHS

$$2\cosh^2 t = 2\left(\frac{1}{4}\left(e^t+e^{-t}\right)^2\right)$$
$$=\frac{1}{2}\left(e^{2t}+2e^te^{-t}+e^{-2t}\right)$$
$$=\frac{1}{2}\left(e^{2t}+2+e^{-2t}\right)$$
$$=1+\frac{1}{2}\left(e^{2t}+e^{-2t}\right)$$
$$=1+\cosh 2t$$

(b) (i)

$$x=2\sinh t \Rightarrow \frac{dx}{dt}=2\cosh t$$
$$y=\cosh^2 t \Rightarrow \frac{dy}{dt}=2\cosh t\sinh t$$

so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=4\cosh^2 t+4\cosh^2 t\sinh^2 t$$
$$=4\cosh^2 t(1+\sinh^2 t)$$
$$=4\cosh^2 t(\cosh^2 t)$$
$$=4\cosh^4 t$$

(ii) Length of curve is given by

$$s=\int_0^{\frac{1}{2}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$=\int_0^{\frac{1}{2}}\sqrt{4\cosh^4 t}\,dt$$
$$=\int_0^{\frac{1}{2}}2\cosh^2 t\,dt$$
$$=\int_0^{\frac{1}{2}}(1+\cosh 2t)\,dt$$
$$=\left[t+\frac{\sinh 2t}{2}\right]_0^{\frac{1}{2}}$$
$$=\frac{1}{2}+\frac{\sinh 1}{2}$$
$$=\frac{1}{2}(1+\sinh 1)$$

(c)

$$y=\cosh^2 t=1+\sinh^2 t=1+\left(\frac{x}{2}\right)^2$$
$$y=\frac{x^2}{4}+1$$

Question 5

Part 1

Assume true for n=k

$$u_k=3\left(\frac{2}{3}\right)^k-1$$

and substitute this into given expression for sequence

$$3u_{k+1}=2\left(3\left(\frac{2}{3}\right)^k-1\right)-1$$
$$u_{k+1}=2\left(\frac{2}{3}\right)^k-1$$
$$3\left(\frac{2}{3}\right)^{k+1}-1$$

Part 2

Now test a specific value of n :- set $n=1$

$$u_1=3\left(\frac{2}{3}\right)^1-1=1$$

which is equal to the stated value for u_1. Therefore true for all values, from 'Part 1'.

Question 6

(a) (i)

$$(\cos\theta+i\sin\theta)^4+(\cos\theta-i\sin\theta)^4$$
$$=(\cos 4\theta+i\sin 4\theta)+(\cos 4\theta-i\sin 4\theta)$$
$$=2\cos 4\theta$$

(ii)

$$(\cot\theta+i)^4+(\cos\theta-i)^4$$
$$=\frac{1}{\sin^4\theta}(\cos\theta+i\sin\theta)^4+\frac{1}{\sin^4\theta}(\cos\theta-i\sin\theta)^4$$

for $\theta\neq r\pi$

$$=\frac{2\cos 4\theta}{\sin^4\theta}$$

using result from (i)

(b) $\cot\theta$ is a root when

$$\cos 4\theta=0$$

i.e. when

$$4\theta=2n\pi\pm\frac{\pi}{2}\quad n=0,1,...$$
$$\theta=\frac{n\pi}{2}\pm\frac{\pi}{8}$$
$$\theta=\pm\frac{\pi}{8},\frac{\pi}{2}\pm\frac{\pi}{8}$$
$$\theta=\pm\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8}$$

so roots are

$$\theta=\pm\cot\frac{\pi}{8},\cot\frac{3\pi}{8},\cot\frac{5\pi}{8}$$

(c)

$$(z+1)^4 = z^4+4iz^3+\frac{(4)(3)}{2!}i^2z^2+\frac{(4)(3)(2)}{3!}i^3z+\frac{(4)(3)(2)(1)}{4!}i^4$$

$$(z-1)^4 = z^4-4iz^3+\frac{(4)(3)}{2!}(-i)^2z^2+\frac{(4)(3)(2)}{3!}(-i)^3z+\frac{(4)(3)(2)(1)}{4!}(-i)^4$$

Adding and simplifying

$$2z^4 - 12z^2 + 2 = 0$$

$$z^4 - 6z^2 + 1 = 0$$

(d) Roots of above expression in (c)

$$z^2 = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2}$$

$$= \frac{6 - \sqrt{32}}{2}$$

$$= 3 \pm 2\sqrt{2}$$

comparing with answer from (b), and noting that

$$\cot\frac{\pi}{8} > 1$$

then

$$\cot^2\frac{\pi}{8} = 3 + 2\sqrt{2}$$

Pure 5 - MAP5, 17 January 2003

Question 1

(a)

$$\lim_{x\to\infty} e^{-4x}\ dx = 0$$

therefore

$$\lim_{x\to\infty} x^k e^{-4x}\ dx = 0$$

because the limit of a product of functions is the product of their individual limits

(b)

$$\int_0^\infty xe^{-4x}dx = -\left[\frac{xe^{-4x}}{4}\right]_0^\infty + \frac{1}{4}\int_0^\infty x^{-4x}\ dx$$

$$= (0) - \frac{1}{16}\left[e^{-4x}\right]_0^\infty$$

$$= -\frac{1}{16}\{(0) - (1)\}$$

$$= \frac{1}{16}$$

(c)

$$\int_0^\infty x^2e^{-4x}dx = -\left[\frac{x^2e^{-4x}}{4}\right]_0^\infty + \frac{1}{4}\int_0^\infty 2xe^{-4x}\ dx$$

$$= \frac{1}{2}\int_0^\infty xe^{-4x}\ dx$$

this is half the integral from (b), so solution to the integral is

$$\frac{1}{32}$$

Question 2

(a) sketch

(b) (i) Area

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\alpha} r^2 d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\alpha} \frac{1}{\theta^2}d\theta$$

$$= \frac{1}{2}\left[-\frac{1}{\theta}\right]_{\frac{\pi}{4}}^{\alpha}$$

$$= \frac{1}{2}\left(-\frac{1}{\alpha} + \frac{4}{\pi}\right)$$

$$= \frac{2}{\pi} - \frac{1}{2\alpha}$$

(ii) Repeating the above integral between 2π and $\frac{\pi}{4}$ gives an area of

$$\frac{2}{\pi} - \frac{1}{4\pi}$$

so if OP bisects this area

$$\frac{2}{\pi} - \frac{1}{2\alpha} = \frac{1}{2}\left(\frac{2}{\pi} - \frac{1}{4\pi}\right)$$

$$\frac{2}{\pi} - \frac{1}{2\alpha} = \frac{1}{\pi} - \frac{1}{8\pi}$$

$$\frac{1}{2\alpha} = \frac{2}{\pi} - \frac{1}{\pi} + \frac{1}{8\pi}$$

$$\frac{1}{2\alpha} = \frac{16 - 8 + 1}{8\pi}$$

$$\frac{1}{2\alpha} = \frac{9}{8\pi}$$

$$2\alpha = \frac{8\pi}{9}$$

$$\alpha = \frac{4\pi}{9}$$

Question 3

(a) Improper integral because

$$|f(x)| \to \infty \quad as \quad |x| \to 2$$

(b)

$$\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 + \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^0$$

$$= \left(\sin^{-1}(1) - \sin^{-1}(-1)\right)$$

$$= \left(\frac{\pi}{2} - (-\frac{\pi}{2})\right)$$

$$= \pi$$

Question 4

(a) (i)

$$y = (x^2+1)y$$

$$\frac{dy}{dx} = (x^2+1)\frac{dy}{dx} + y(2x)$$

$$\frac{dy}{dx} - (x^2+1)\frac{dy}{dx} = 2xy$$

$$-x^2\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = -\frac{2y}{x}$$

(ii)

$$\frac{d}{dx}\left(\left(x^2+1\right)y\right) = 6x^2+2$$
$$\left(x^2+1\right)y = 2x^3+2x+c$$
$$y = \frac{\left(2x^3+2x+c\right)}{x^2+1}$$

(b) from (a)(ii) the soln for y is

$$y = \frac{dz}{dx} = \frac{\left(2x^3+2x+c\right)}{x^2+1}$$

when $x = 0$, $\frac{dz}{dx} = 1$

$$\Rightarrow 1 = c$$

producing

$$\int dz = \left(\frac{\left(2x^3+2x+1\right)}{x^2+1}\right)dx$$
$$\int dz = \left(2x + \frac{1}{x^2+1}\right)dx$$
$$z = x^2 + \tan^{-1}x + D$$

when $x = 0, z = 2$, so

$$2 = D \Rightarrow D = 2$$

solution is

$$z = x^2 + \tan^{-1}x + 2$$

Question 5

(a)

$$m = \frac{-2 \pm \sqrt{4-20}}{2}$$
$$= -1 \pm 2i$$

(b)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10$$

Auxiliary Equation is as given in (a), so CF is

$$y = e^{-x}(A\cos 2x + B\sin 2x)$$

Particular Integral : try $y_0 = k \Rightarrow y_0' = 0$

$$5k = 10 \Rightarrow k = 2$$

So GS is

$$y = e^{-x}(A\cos 2x + B\sin 2x) + 2$$

When $x = 0, y = 0$

$$0 = A + 2 \Rightarrow A = -2$$

Differentiating y

$$\frac{dy}{dx} = e^{-x}(-2\sin 2x + 2B\cos 2x) - e^{-x}(-2\cos 2x + B\sin 2x)$$

When $x = 0$, $\frac{dy}{dx} = 0$

$$0 = (2B) + 2 \Rightarrow B = -1$$

So solution is

$$y = e^{-x}(-2\cos 2x - \sin 2x) + 2$$

Question 6

(a)

$$y_1 = 0.5 + 0.1\frac{1 \times 0.5}{\sqrt{1^2+0.5^2}}$$
$$= 0.5447 \text{ to 4 d.p.}$$

(b) (i)

$$k_1 = 0.1\frac{1 \times 0.5}{\sqrt{1^2+0.5^2}} = 0.04472...$$

$$k_2 = 0.1\frac{1.1 \times 0.5447}{\sqrt{1.1^2+0.5447^2}} = 0.04881......$$

$$y_1 = 0.5 + \frac{1}{2}(0.04472 + 0.04881) = 0.5468 \text{ to 4 d.p.}$$

(ii)

$$k_1 = 0.1\frac{1.1 \times 0.5468}{\sqrt{1.1^2+0.5468^2}} = 0.04896...$$

$$k_2 = 0.1\frac{1.2 \times (0.5468 + 0.04896)}{\sqrt{1.2^2 + (0.5468+0.04896)^2}} = 0.05336......$$

$$y_2 = 0.5468 + \frac{1}{2}(0.04896 + 0.05336) = 0.598 \text{ to 3 d.p.}$$

Pure 6 - MAP6, 17 January 2003

Question 1

(a) Since clockwise is considered a negative rotation

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

(b) the first vector gives

$$\begin{bmatrix} 1 & \alpha \\ 2 & \beta \end{bmatrix}$$

where α and β can be any value, while the second vector would require

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Question 2

(a)

$$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$$

(b)

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = (\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 4 & -1 \end{bmatrix}$$

Question 3

$$(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}+\mathbf{b}+5\mathbf{c})$$

$$= \mathbf{a}\times\mathbf{a} + \mathbf{a}\times\mathbf{b} + 5\mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{b} + 5\mathbf{b}\times\mathbf{c}$$

$$= \mathbf{a}\times\mathbf{b} + 5(-2\mathbf{j}) - \mathbf{a}\times\mathbf{b} + 5\mathbf{i}$$

$$= 5\mathbf{i} - 10\mathbf{j}$$

Question 4

(a)

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)((2-\lambda)(3-\lambda) = 0$$

therefore

$$\lambda = 1, 2, 3$$

(b)

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which corresponds to $\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

which corresponds to $\lambda = 3$

(c) Using $\lambda = 2$

$$\begin{aligned} -x+y+3z &= 0 \\ 5z &= 0 \\ z &= 0 \end{aligned}$$

$$\Rightarrow z = 0, \quad x = y$$

Therefore required vector

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(d)

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(e) Using eigenvalues above

$$\mathbf{r} = \alpha\mathbf{v}_1 + 3\beta\mathbf{v}_2 + 2\gamma\mathbf{v}_3$$

Question 5

(a)

$$\begin{vmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= x^2(1-2) - x(1-4) + (2-4)$$

$$= -x^2 + 3x - 2$$

$$= -(x^2 - 3x + 2)$$

$$-(x-2)(x-1)$$

(b)

$$\begin{vmatrix} x & 0 & 2 \\ 0 & x & 9 \\ 0 & 1 & x \end{vmatrix}$$

$$= x(x^2 - 9)$$

$$= x(x+3)(x-3)$$

Since $det(\mathbf{AB}) = \det\mathbf{A} \times \det\mathbf{B}$

$$det(\mathbf{AB}) = -\mathrm{x}(\mathrm{x}+3)(\mathrm{x}-3)(\mathrm{x}-2)(\mathrm{x}-1)$$

Question 6

(a)

$$\begin{bmatrix} 3 & -1 & -5 & 5 \\ 2 & 1 & -5 & 10 \\ 1 & 1 & a & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -5 & 5 \\ 0 & -1 & -5-2a & -4 \\ 1 & 1 & a & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -5 & 5 \\ 0 & -1 & -5-2a & -4 \\ 0 & 4 & 3a+5 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -5 & 5 \\ 0 & -1 & -5-2a & -4 \\ 0 & 0 & -5a-15 & 0 \end{bmatrix}$$

$$(-5a - 15)z = 0 \Rightarrow z = 0 \text{ when } a \neq -3$$

(b) (i) When $a = -3$

$$(-5a - 15)z = 0 \Rightarrow z = k \quad \text{i.e. z is arbitrary}$$

and

$$-y + (-5 - 2(-3))t = -4$$

$$-y + t = -4 \Rightarrow y = 4 + t$$

and

$$3x - (4 + t) - 5t = 5$$

$$3x = 6t + 9 \Rightarrow x = 2t + 3$$

(ii) the planes will intersect in a common line

Question 7

(a)

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= \mathbf{i}(2) - \mathbf{j}(-3) + \mathbf{k}(-1)$$

$$= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

(b) (i)

$$\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - s\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} t-1 \\ 3-s \\ 2+2t-3s \end{bmatrix}$$

(ii)

$$\begin{bmatrix} t-1 \\ 3-s \\ 2+2t-3s \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$t - 1 = 2\lambda \Rightarrow t = 2\lambda + 1$$

$$3 - s = 3\lambda \Rightarrow s = 3 - 3\lambda$$

and

$$2 + 2t - 3s = -\lambda$$

$$2 + 2(2\lambda + 1) - 3(3 - 3\lambda) = -\lambda$$

$$2 + 4\lambda + 2 - 9 + 9\lambda + \lambda = 0$$

$$14\lambda - 5 = 0$$

$$\lambda = \frac{5}{14}$$

(c)

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0$$

(d) Finding the modulus of $\vec{PQ}$

$$\frac{5}{14}\sqrt{14}$$

Mechanics 1 - MAM1, 13 January 2003

Question 1

$$2m(2) - m(6) = -2m(3V) + m(V)$$
$$-2m = -5mV$$
$$V = 0.4ms^{-1}$$

Question 2

(a) sketch

(b)

$$v^2 + 1^2 = 1.25^2$$
$$v^2 = \frac{25}{16} - 1 = \frac{9}{16}$$
$$v = \frac{3}{4}ms^{-1}$$

(c)

$$\sin\theta = \frac{1}{1.25} = 0.8$$
$$\theta = 53.1^\circ$$

Question 3

(a)

$$v = u + at$$
$$v = 0 + 0.8 \times 3$$
$$v = 2.4ms^{-1}$$

(b) sketch

(c) Distance traveled to 3 secs

$$s = \frac{1}{2}(2.4) \times 3 = 3.6m$$

Distance traveled between 3 and 7 secs

$$s = 2.4 \times 4 = 9.6m$$

Total distance

$$= 3.6 + 9.6 = 13.2m$$

Average Speed

$$= \frac{13.2}{7} = 1.89ms^{-1}$$

Question 4

(a) (i) Let T be the tension in the string.

Resolving vertically

$$T = mg$$

which is the answer

(ii) Resolving horizontally

$$\mu R \geq T$$
$$\mu(2mg) \geq T$$
$$2\mu mg \geq mg$$
$$\mu \geq \frac{1}{2}$$

(b) Vertically

$$mg - T = ma$$

Horizontally

$$T = 2ma$$

Substituting

$$T = 2(mg - T)$$
$$3T = 2mg$$
$$T = \frac{2mg}{3}$$

Question 5

(a) Resolving horizontally

$$6 = 8\cos 60 + a\cos 60$$
$$6 = 4 + \frac{1}{2}a$$
$$a = 4$$

(b) Resolving vertically upwards

$$R = 4\cos 30 - 8\cos 30$$
$$= -4\frac{\sqrt{3}}{2}$$
$$= -2\sqrt{3}$$

so R is $2\sqrt{3}$ N downwards

Question 6

(a) Distance traveled for $t \geq 0$

$$s = \int (2t\mathbf{i} + 4\mathbf{j})\mathrm{dt}$$
$$= t^2\mathbf{i} + 4\mathrm{t}\mathbf{j}$$

So position vector at time t

$$\mathbf{r} = 2\mathbf{j} + (\mathrm{t}^2\mathbf{i} + 4\mathrm{t}\mathbf{j})$$
$$= t^2\mathbf{i} + (4\mathrm{t} + 2)\mathbf{j}$$

(b)

$$\vec{CD} = 16\mathbf{i} + 18\mathbf{j} - (4\mathbf{i} + 10\mathbf{j})$$
$$= 12\mathbf{i} + 8\mathbf{j}$$

Question 7

(a) During flight

$$\mathbf{v} = \begin{bmatrix} 7t \\ 7t - gt \end{bmatrix}$$

Integrating to find position vector (constants of integration are zero)

$$\mathbf{r} = \begin{bmatrix} 7t \\ 7t - \frac{1}{2}gt^2 \end{bmatrix}$$

(b)

$$y = 7t - \frac{1}{2}gt^2$$
$$x = 7t$$

so

$$y = x - \frac{1}{2}g\left(\frac{x}{7}\right)^2$$

$$y = x - \frac{1}{2}g\left(\frac{x}{7}\right)^2$$

$$y = x - 0.1x^2$$

(c) When $x = 4$

$$y = 4 - 0.1(4)^2$$
$$y = 2.4$$

so ball passes over the wire

(d) Set $y = 0$

$$0 = x - 0.1x^2$$
$$x(1 - 0.1x) = 0$$

so

$$x = 0, 10$$

so answer is 10 m

(e) Max speed *(at t=0)*

$$= \sqrt{7^2 + 7^2}$$
$$= \sqrt{98} = 9.90ms^{-1}$$

Mechanics 2 - MAM2, 22 January 2003

Question 1

$$P = \mathbf{F}.\mathbf{v} = (6,8).(9,12) = 54 + 96 = 150\text{W}$$

Question 2

$$M\bar{x} = m_{AB}x_{AB} + m_{CDEF}x_{CDEF}$$

$$M\bar{x} = (0.25)(0.4) + (1)(0.95)$$

$$\bar{x} = \frac{(0.25)(0.4) + (1)(0.95)}{1.25}$$

$$= \frac{1.05}{1.25}$$

$$= 0.84m$$

Question 3

(a) (i) EPE

$$= \frac{1}{2}kx^2 = \frac{1}{2}10 \times 0.5^0.5 = 1.25J$$

(ii) KE gained + EPE gained = PE lost

$$\frac{1}{2}mv^2 + 1.25 = mgh$$

$$\frac{1}{2}(0.1)v^2 + 1.25 = 0.1g(1.5)$$

$$\frac{1}{2}(0.1)v^2 = 0.15g - 1.25$$

$$v^2 = 20(0.15g - 1.25) = 4.4$$

$$v = 2.098ms{-}1 \text{ to 3 d.p. } \approx 2.10ms^{-1}$$

(b) Rebound Speed = $\frac{3}{5}\times$ Approach Speed

$$= \frac{3}{5}2.1$$

$$= 1.26ms^{-1}$$

(c)

$$v^2 = u^2 - 2gs$$

$$0 = 1.26^2 - 2gs$$

$$s = \frac{1.26^2}{2g}$$

$$s = 0.081m$$

Question 4

(a) Conservation of Momentum

$$1200(43.2) = 1200v_A + 1500v_B$$

Coefficient of Restitution

$$\frac{1}{4} \times 43.2 = (v_B - v_A)$$

$$v_A = v_B - \frac{1}{4}(43.2)$$

Substituting this in top eqn.

$$1200(43.2) = 1200(v_B - \frac{1}{4}(43.2)) + 1500v_B$$

$$2700v_B = 1200(43.2) + 1200(\frac{1}{4})(43.2)$$

$$v_B = \frac{5(1200)(43.2)}{4 \times 2700}$$

$$v_B = 24ms^{-12}$$

Therefore

$$v_A = 24 - \frac{1}{4}(43.2)$$

$$= 13.2ms^{-1}$$

(b) Impulse of A on B = Change in momentum of B

$$= 1500 \times \frac{24 \times 1000}{3600}$$

$$= 10000Ns$$

(c) (i) sketch of F against t, of triangular shape, starting from origin

(ii) equate the area of this triangle to 10 000

$$\frac{1}{2}F(0.01) = 10000$$

$$F = 2 \times 10^6 N$$

(iii) Force will probably not vary in such a 'nice' linear way

Question 5

(a) Resolving horizontally

$$T\sin\theta = ml\sin\theta\omega^2$$
$$T = ml\omega^2$$

(b) Resolving vertically

$$T\cos\theta = mg$$
$$ml\omega^2\cos\theta = mg$$
$$\cos\theta = \frac{g}{l\omega^2}$$

(c) (i) From (a), if $T \le 16N$

$$ml\omega^2 \le 16$$
$$(0.1)(0.4)\omega^2 \le 16$$
$$\omega^2 \le 400$$
$$\omega < 20\ rad/sec$$

(ii) From (b)

$$\cos\theta = \frac{g}{l\omega^2}$$
$$\theta = \cos^{-1}\frac{g}{l\omega^2}$$

this will be a maximum when $\omega = 20$

$$\theta = \cos^{-1}\frac{9.8}{(0.4)(20)^2}$$
$$= 86^\circ \text{ to nearest degree}$$

Question 6

(a) Equating KE gained to PE lost

$$\frac{1}{2}mu^2 = mg(r)$$
$$u^2 = 2gr$$

(b) (i) KE at P = KE at B plus PE lost between B and P

$$\frac{1}{2}mv^2 = \frac{1}{2}m(2gr) + mgr\sin\theta$$
$$v^2 = 2gr + 2gr\sin\theta$$
$$v^2 = 2gr(1+\sin\theta)$$

(ii) diagram showing Weight downwards and Reaction at right angles to the surface

(iii) Resolving along R and equating to central force

$$R - mg\sin\theta = \frac{mv^2}{r}$$
$$R - mg\sin\theta = \frac{m(2gr(1+\sin\theta))}{r}$$
$$R - mg\sin\theta = \frac{m(2gr(1+\sin\theta))}{r}$$
$$R = mg(\sin\theta 2 + 2\sin\theta)$$
$$R = mg(2 + 3\sin\theta)$$

(c) Just before C, with $\theta = 90^\circ$

$$R = mg(2+3) = 5mg$$

After C

$$R = mg$$

Therefore a change of magnitude $4mg$

(d) At C

$$v^2 = 2gr(1+1) = 4gr$$

Using

$$v^2 = u^2 + 2as$$
$$0 = 4gr + 2a(4r)$$
$$a = -\frac{g}{2}$$

Mechanics 3 - MAM3, 28 January 2003

Question 1

(a) Maximum angular speed

$$= \frac{1100}{60} \times 2\pi$$

$$= 115 \text{ to 3 s.f.}$$

(b) Angular acceleration

$$= \frac{115}{8} = 14.4 \, rad/sec^2$$

(c)

$$T = I\alpha$$

where α is angular acceleration

$$T = 20 \times 14.4$$

$$= 288Nm$$

Question 2

(a) sketch - horizontal Reaction at R, Reaction perpendicular to pole at S, and weight W

(b) Resolving vertically

$$S \cos\theta = W$$

Taking moments about A

$$W \cos\theta(2.5) = S\frac{2}{\cos\theta}$$

$$\frac{W}{S}\cos\theta(2.5) = \frac{2}{\cos\theta}$$

$$\cos\theta\cos\theta(2.5) = \frac{2}{\cos\theta}$$

$$\cos^3\theta = \frac{2}{2.5}$$

$$\theta = \cos^{-1}\sqrt[3]{\frac{2}{2.5}}$$

$$= 21.8^\circ \text{ to 3 s.f.}$$

Question 3

(a) Moment of Inertia

$$= \frac{4}{3}(2m)(3a)^2$$

$$= 24ma^2$$

(b) Instantaneously at impact, angular momentum

$$= mvr = mv(3a) = 3mva$$

After impact, angular momentum

$$= I_{(plate)}\omega + m(3a)^2\omega$$

$$= 24ma^2\omega + 9ma^2\omega$$

$$= 33ma^2\omega$$

so using Conservation of Angular Momentum

$$3mva = 33ma^2\omega$$

$$\omega = \frac{v}{11a} rad \; s^{-1}$$

Question 4

(a)

$$T_1 \cos 60 = 100$$

$$T_1 = 200N$$

and rod DC is in tension

(b) (i)

$$T_1 = T_2 \cos 30$$

$$T_2 = \frac{2 \times 200}{\sqrt{3}}$$

$$T_2 = 231N$$

(ii)

$$T_3 = T_2 \cos 60$$

$$= 231 \times \frac{1}{2}$$

$$= 115N (\text{to 3 s.f., if using exact value from (b)(i)}$$

(c) Resolving Horizontally (+ve to right)

$$T_3 \cos 60 - T_4$$

$$T_3 \cos 60 - T_1 \cos 30$$

$$= 115\left(\frac{1}{2}\right) - 200.\frac{\sqrt{3}}{2}$$

$$= -116N$$

Resolving Vertically (+ve upwards)

$$= T_3 \cos 30$$

$$= 115.\frac{\sqrt{3}}{2}$$

$$= 100N$$

Question 5

(a)

	Weight	*Distance of C of G from O*
Cylinder	$3\pi r^3\rho$	$\frac{3r}{2}$
Cone	$\frac{1}{3}\pi r^3\rho$	$\frac{11r}{4}$
Wholefigure	$\frac{8}{3}\pi r^3\rho$	$\bar{x}$

so

$$\frac{8}{3}\pi r^3\rho\bar{x} = 3\pi r^3\rho \times \frac{3r}{2} - \frac{1}{3}\pi r^3\rho \times \frac{11}{12}$$

$$\frac{8\bar{x}}{3} = \frac{9r}{2} - \frac{11r}{12}$$

$$32\bar{x} = 54r - 11r$$

$$\bar{x} = \frac{43r}{32}$$

(b) Body will topple when C of G moves to the left of bottom left hand corner, i.e. when

$$\tan\theta > \frac{r}{\bar{x}}$$

$$\theta > \tan^{-1}\frac{r}{43r/32}$$

$$> \tan^{-1}\frac{32}{43}$$

$$> 36.7^\circ$$

Question 6

(a) (i) Loss in PE = Gain in KE

$$mga - mga\cos\theta = \frac{1}{2}\left(\frac{4}{3}ma^2\right)\dot{\theta}^2$$

$$\dot{\theta}^2 = \frac{6ga(1-\cos\theta)}{4a^2}$$

$$\dot{\theta}^2 = \frac{3g}{2a}(1-\cos\theta)$$

(ii) Differentiating expression from (a)

$$2\dot{\theta}\ddot{\theta} = \frac{3g(\sin\theta)}{2a}\dot{\theta}$$

$$\ddot{\theta} = \frac{3g(\sin\theta)}{4a}$$

(b) (i) Resolving along rod and equating to centripetal force

$$mg\cos\theta - H_{AB} = ma\dot{\theta}^2$$

$$H_{AB} = mg\cos\theta - ma\frac{3g(1-\cos\theta)}{2a}$$

$$H_{AB} = mg\left(\cos\theta - \frac{3(1-\cos\theta)}{2}\right)$$

$$H_{AB} = \frac{mg}{2}(5\cos\theta - 3)$$

(ii) Resolving perpendicular to rod and equating to angular acceleration

$$mg\sin\theta - H_\perp = ma\ddot{\theta}$$

$$H_\perp = mg\sin\theta - ma\frac{3g(\sin\theta)}{4a}$$

$$H_\perp = mg\left(\sin\theta - \frac{3(\sin\theta)}{4}\right)$$

$$H_\perp = \frac{mg}{4}(\sin\theta)$$

(c) When B is vertically below A, $\theta = \pi \Rightarrow \sin\theta = 0, \cos\theta = -1$

so

$$H_\perp = 0$$

$$H_{AB} = \frac{mg}{2}(-5-3)$$

$$= -4mg$$

Mechanics 4 - MAM4, 30 January 2003

Question 1

(a)

$$a = 2x - \frac{1}{4}x^2$$

$$v\frac{dv}{dx} = 2x - \frac{1}{4}x^2$$

$$\int v dv = \int \left(2x - \frac{1}{4}x^2\right) dx$$

$$\frac{v^2}{2} = x^2 - \frac{1}{12}x^3 + C$$

When x=0, v=0

$$\Rightarrow C = 0$$

so

$$v^2 = 2x^2 - \frac{1}{6}x^3$$

(b) When v=0

$$0 = 2x^2 - \frac{1}{6}x^3$$

$$x^2\left(\frac{1}{6}x - 2\right) = 0$$

$$\Rightarrow x = 12m \text{ when it stops for the first time}$$

Question 2

(a) <u>At time t</u>

$$v = v; \quad m = M + rt$$

<u>At time $t + \delta t$</u>

$$v = v + \delta t; \quad m = M + rt + r\delta t$$

$$(M + rt + r\delta t)(v + \delta v) - (M + rt)v = 0$$

$$Mv + M\delta v + rtv + rt\delta v + rv\delta t + r\delta t\delta v - Mv + rtv = 0$$

$$M\delta v + rt\delta v + rv\delta t + r\delta t\delta v = 0$$

dividing by δt, taking the limits ignoring higher order terms

$$M\frac{dv}{dt} + rt\frac{dv}{dt} + rv = 0$$

$$(M + rt)\frac{dv}{dt} + rv = 0$$

(b)

$$\int \frac{dv}{v} = -\int \frac{rdt}{M + rt}$$

$$\ln v = -\ln(M + rt) + C$$

when $t = 0, \quad v = u$

$$\ln u = -\ln M + C$$

$$C = \ln u + \ln M = \ln uM$$

so

$$\ln v = \ln uM - \ln(M + rt)$$

$$= \ln \frac{uM}{(M + rt)}$$

$$v = \frac{uM}{(M + rt)}$$

Question 3

(a)

$$T = \lambda x$$

so

$$T = \frac{2mg}{l}(r - l)$$

(c) There is no transverse force, so

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

so

$$r^2\dot{\theta} = k$$

When $r = l, r\dot{\theta} = \sqrt{gl}$ (as given in question)

$$k = l\sqrt{gl}$$

so

$$r^2\dot{\theta} = l\sqrt{gl}$$

$$\dot{\theta} = \frac{\sqrt{gl^3}}{r^2}$$

(c)

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -T$$

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{2mg}{l}(r - l)$$

$$\ddot{r} = r\dot{\theta}^2 - \frac{2g}{l}(r - l)$$

$$\ddot{r} = r\left(\frac{\sqrt{gl^3}}{r^2}\right)^2 - \frac{2g}{l}(r - l)$$

$$\ddot{r} = \frac{gl^3}{r^3} - \frac{2g}{l}(r - l)$$

(d) ignored any air resistance
modelled ball as a particle
assumed spring obeyed Hooke's law

Question 4

(a) Auxiliary equation

$$m^2 + 121 = 0$$

$$m = \pm 11i$$

so Complementary Function is

$$x = A\sin 11t + B\cos 11t$$

Particular Integral

let

$$x = C\sin\omega t + D\cos\omega t$$
$$\dot{x} = C\omega\cos\omega t - D\omega\sin\omega t$$
$$\ddot{x} = -C\omega^2\sin\omega t - D\omega^2\cos\omega t$$

so

$$-C\omega^2\sin\omega t - D\omega^2\cos\omega t + 121(C\sin\omega t + D\cos\omega t)$$
$$= 3\sin\omega t$$

Comparing coefficients of $\sin\omega t$

$$-C\omega^2 + 121C = 3$$
$$C = \frac{3}{121-\omega^2}$$

Comparing coefficients of $\cos\omega t$

$$-D\omega^2 + 121D = 0 \Rightarrow D = 0$$

So General Solution is

$$x = A\sin 11t + B\cos 11t + \frac{3}{121-\omega^2}\sin\omega t$$

(b) (i) 11

(ii) let

$$x = t(C\sin 11t + D\cos 11t)$$
$$\dot{x} = t(11C\cos 11t - 11D\sin 11t) + (C\sin 11t + D\cos 11t)$$
$$\ddot{x} = t(-121C\sin 11t - 121D\cos 11t) +$$
$$11C\cos 11t - 11D\sin 11t + 11C\cos 11t - 11D\sin 11t$$

so

$$t(-121C\sin 11t - 121D\cos 11t) + 11C\cos 11t -$$
$$11D\sin 11t + 11C\cos 11t$$
$$-11D\sin 11t + 121(t(C\sin 11t + D\cos 11t)) = 3\sin 11t$$

comparing coefficients of $\sin 11t$

$$-121tC - 11D - 11D + 121tC = 3$$
$$-22D = 3$$
$$D = -\frac{3}{22}$$

comparing coefficients of $\cos 11t$

$$-121tD + 11C + 11C + 121tD = 0$$
$$22C = 0 \Rightarrow C = 0$$

so Particular Integral is

$$x = t\left(-\frac{3}{22}\cos 11t\right)$$

Question 5

(a)

$$\ddot{x} = 3\dot{x} - 2\dot{y}$$
$$\ddot{x} = 3\dot{x} - 2(2y - x)$$
$$\ddot{x} = 3\dot{x} - 2([3x - \dot{x}] - x)$$
$$\ddot{x} - 5\dot{x} + 4x = 0$$

(b) Auxiliary Equation

$$m - 5m + 4 = 0$$
$$(m-4)(m-1) = 0$$
$$m = 4, 1$$

so General Solution is

$$x = Ae^{4t} + Be^{t}$$

differentiating

$$\dot{x} = 4Ae^{4t} + Be^{t}$$

When t=0, x =5000

$$5000 = A + B$$

Using the eqn for $\dot{x}$ above and the form given in the question

$$4Ae^{4t} + Be^{t} = 3x - 2y$$

Considering t=0, when x=5000 and y=2000

$$4A + B = 3(5000) - 2(2000)$$
$$4A + (5000 - A) = 11000$$
$$3A = 6000$$
$$A = 2000$$

and therefore

$$5000 = 2000 + B$$
$$B = 3000$$

Solution is

$$x = 2000e^{4t} + 3000e^{t}$$

(c)

$$y = 3000e^{t} - 1000e^{4t}$$
$$\dot{y} = 3000e^{t} - 4000e^{4t}$$

From the question

$$\dot{y} = 2y - x$$
$$= 2(3000e^{t} - 1000e^{4t}) - 2000e^{4t} - 3000e^{t}$$
$$= 3000e^{t} - 4000e^{4t}$$

which is the same as above - so the given formula is verified

(d) Set $y = 0$

$$0 = 3000e^{t} - 1000e^{4t}$$
$$1000e^{3t} = 3000$$
$$e^{3t} = 3$$
$$3t = \ln 3$$
$$t = \frac{\ln 3}{3}$$
$$t = 0.366s$$