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Exam Questions, Newcastle University

(which Newcastle University claim are "not even remotely similar")

<table>
<tr><th>Second Year</th><th>Third Year</th></tr>
<tr><td colspan="2">Given Equation</td></tr>
<tr><td>

$$xt^3\frac{\partial^2 u}{\partial x^2} - x^3 t\frac{\partial^2 u}{\partial t^2} - t^3\frac{\partial u}{\partial x} + x^3\frac{\partial u}{\partial t} = 0 \qquad (1)$$

</td><td>

$$t\frac{\partial^2 u}{\partial x^2} + (t-x)\frac{\partial^2 u}{\partial x \partial t} - x\frac{\partial^2 u}{\partial t^2} = \frac{t-x}{t+x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right) + 4(x+t)^2(x-t) \qquad (2)$$

</td></tr>
<tr><td colspan="2">To find the characteristic curves</td></tr>
<tr><td>

$$"a" = xt^3 \quad "b" = 0 \quad "c" = -x^3 t$$

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{c}$$

$$= \frac{0 \pm \sqrt{0 - (xt^3)(-x^3 t)}}{-x^3 t}$$

$$= \frac{\pm\sqrt{x^4 t^4}}{-x^3 t} = \frac{\pm x^2 t^2}{-x^3 t} = \pm\frac{t}{x}$$

One characteristic is given by

$$\frac{dx}{dt} = -\frac{t}{x} \Rightarrow \int x dx = -\int t dt \Rightarrow x^2 + t^2 = \text{constant}$$

The other characteristic is given by

$$\frac{dx}{dt} = \frac{t}{x} \Rightarrow \int x dx = \int t dt \Rightarrow x^2 - t^2 = \text{constant}$$

</td><td>
Unable to do this, because it is first - year material, and I was not at Newcastle during the first year.
</td></tr>
<tr><td colspan="2">Therefore the characteristics are given by</td></tr>
<tr><td>

$$\zeta = x^2 + t^2, \qquad \eta = x^2 - t^2$$

</td><td>

$$\zeta = x - t, \qquad \eta = x^2 + t^2$$

</td></tr>
<tr><td colspan="2">Derivatives of these characteristics</td></tr>
<tr><td>

$$\begin{array}{|cc|cc|} \multicolumn{2}{|c|}{\zeta = x^2 + t^2} & \multicolumn{2}{c|}{\eta = x^2 - t^2} \\ \hline \zeta_x = 2x & \zeta_t = 2t & \eta_x = 2x & \eta_t = -2t \\ \zeta_{xx} = 2 & \zeta_{tt} = 2 & \eta_{xx} = 2 & \eta_{tt} = -2 \\ \multicolumn{2}{c}{\zeta_{xt} = 0} & \multicolumn{2}{c}{\eta_{xt} = 0} \end{array}$$

</td><td>

$$\begin{array}{|cc|cc|} \multicolumn{2}{|c|}{\zeta = x - t} & \multicolumn{2}{c|}{\eta = x^2 + t^2} \\ \hline \zeta_x = 1 & \zeta_t = -1 & \eta_x = 2x & \eta_t = 2t \\ \zeta_{xx} = 0 & \zeta_{tt} = 0 & \eta_{xx} = 2 & \eta_{tt} = 2 \\ \multicolumn{2}{|c|}{\zeta_{xt} = 0} & \multicolumn{2}{c|}{\eta_{xt} = 0} \end{array}$$

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<table>
<tr><th>Second Year</th><th>Third Year</th></tr>
<tr><td colspan="2">Introduce ζ and η as new co-ordinates in equation (1) and (2), respectively</td></tr>
<tr><td>$xt^3\left(\zeta_x^2\frac{\partial^2}{\partial\zeta^2}+2\zeta_x\eta_x\frac{\partial^2}{\partial\zeta\partial\eta}+\eta_x^2\frac{\partial^2}{\partial\eta^2}+\zeta_{xx}\frac{\partial}{\partial\zeta}+\eta_{xx}\frac{\partial}{\partial\eta}\right)u$
$-x^3t\left(\zeta_t^2\frac{\partial^2}{\partial\zeta^2}+2\zeta_t\eta_t\frac{\partial^2}{\partial\zeta\partial\eta}+\eta_t^2\frac{\partial^2}{\partial\eta^2}+\zeta_{tt}\frac{\partial}{\partial\zeta}+\eta_{tt}\frac{\partial}{\partial\eta}\right)u$
$-t^3\left(\zeta_x\frac{\partial}{\partial\zeta}+\eta_x\frac{\partial}{\partial\eta}\right)u$
$+x^3\left(\zeta_t\frac{\partial}{\partial\zeta}+\eta_t\frac{\partial}{\partial\eta}\right)u=0$</td><td>$t\left(\zeta_x^2\frac{\partial^2}{\partial\zeta^2}+2\zeta_x\eta_x\frac{\partial^2}{\partial\zeta\partial\eta}+\eta_x^2\frac{\partial^2}{\partial\eta^2}+\zeta_{xx}\frac{\partial}{\partial\zeta}+\eta_{xx}\frac{\partial}{\partial\eta}\right)u$
$(t-x)\left(\zeta_x\zeta_t\frac{\partial^2}{\partial\zeta^2}+(\zeta_x\eta_t+\zeta_t\eta_x)\frac{\partial^2}{\partial\zeta\partial\eta}+\eta_x\eta_t\frac{\partial^2}{\partial\eta^2}+\zeta_{xt}\frac{\partial}{\partial\zeta}+\eta_{xt}\right.$
$-x\left(\zeta_t^2\frac{\partial^2}{\partial\zeta^2}+2\zeta_t\eta_t\frac{\partial^2}{\partial\zeta\partial\eta}+\eta_t^2\frac{\partial^2}{\partial\eta^2}+\zeta_{tt}\frac{\partial}{\partial\zeta}+\eta_{tt}\frac{\partial}{\partial\eta}\right)u$
$=$
$\frac{t-x}{t+x}\left(\zeta_x\frac{\partial}{\partial\zeta}+\eta_x\frac{\partial}{\partial\eta}\right)u$
$+\frac{t-x}{t+x}\left(\zeta_t\frac{\partial}{\partial\zeta}+\eta_t\frac{\partial}{\partial\eta}\right)u$
$+4(x+t)^2(x-t)$</td></tr>
<tr><td colspan="2">Re-arrange</td></tr>
<tr><td>$\left(xt^3\zeta_x^2-x^3t\zeta_t^2\right)u_{\zeta\zeta}$
$+\left(2xt^3\zeta_x\eta_x-2x^3t\zeta_t\eta_t\right)u_{\zeta\eta}$
$+\left(xt^3\eta_x^2-x^3t\eta_t^2\right)u_{\eta\eta}$
$+\left(xt^3\zeta_{xx}-x^3t\zeta_{tt}-t^3\zeta_x+x^3\zeta_t\right)u_\zeta$
$+\left(xt^3\eta_{xx}-x^3t\eta_{tt}-t^3\eta_x+x^3\eta_t\right)u_\eta=0$</td><td>$\left(t\zeta_x^2+(t-x)\zeta_x\zeta_t-x\zeta_t^2\right)u_{\zeta\zeta}$
$+\left(2t\zeta_x\eta_x+(t-x)(\zeta_x\eta_t+\eta_x\zeta_t)-2x\zeta_t\eta_t\right)u_{\zeta\eta}$
$+\left(t\eta_x^2+(t-x)\eta_x\eta_t-x\eta_t^2\right)u_{\eta\eta}$
$+\left(t\zeta_{xx}+(t-x)\zeta_{xt}-x\zeta_{tt}\right)u_\zeta$
$+\left(t\eta_{xx}+(t-x)\eta_{xt}-x\eta_{tt}\right)u_\eta$
$=$
$\frac{t-x}{t+x}\left(\zeta_x+\zeta_t\right)u_\zeta$
$\frac{t-x}{t+x}\left(\eta_x+\eta_t\right)u_\eta$
$+4(x+t)^2(x-t)$</td></tr>
<tr><td colspan="2">Substitute for derivatives of ζ and η</td></tr>
<tr><td>$\left(xt^3(2x)^2-x^3t(2t)^2\right)u_{\zeta\zeta}$
$+\left(2xt^3(2x)(2x)-2x^3t(2t)(-2t)\right)u_{\zeta\eta}$
$+\left(xt^3(2x)^2-x^3t(-2t)^2\right)u_{\eta\eta}$
$+\left(xt^3(2)-x^3t(2)-t^3(2x)+x^3(2t)\right)u_\zeta$
$+\left(xt^3(2)-x^3t(-2)-t^3(2x)+x^3(-2t)\right)u_\eta=0$</td><td>$\left(t(1)^2+(t-x)(1)(-1)-x(-1)^2\right)u_{\zeta\zeta}$
$+\left(2t(1)(2x)+(t-x)((1)(2t)+(2x)(-1))-2x(-1)(2t)\right)u_{\zeta\eta}$
$+\left(t(2x)^2+(t-x)(2x)(2t)-x(2t)^2\right)u_{\eta\eta}$
$+\left(t(0)+(t-x)(0)-x(0)\right)u_\zeta$
$+\left(t(2)+(t-x)(0)-x(2)\right)u_\eta$
$=$
$\frac{t-x}{t+x}\left(1+(-1)\right)u_\zeta$
$\frac{t-x}{t+x}\left(2x+2t\right)u_\eta$
$+4(x+t)^2(x-t)$</td></tr>
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<table>
<tr><th>Second Year</th><th>Third Year</th></tr>
<tr><td colspan="2">Simplifies to normal form</td></tr>
<tr><td>$$(8x^3t^3 + 8x^3t^3)u_{\zeta\eta} = 0$$
$$16x^3t^3u_{\zeta\eta} = 0$$
$$u_{\zeta\eta} = 0$$</td><td>$$(4xt + (t-x)(2t-2x) + 4xt)u_{\zeta\eta} + (2t-2x)u_\eta$$
$$= 2(t-x)u_\eta + 4(x+t)^2(x-t)$$
$$(8xt + 2t^2 - 2xt - 2xt + 2x^2)u_{\zeta\eta} = 4(x+t)^2(x-t)$$
$$(4xt + 2t^2 + 2x^2)u_{\zeta\eta} = 4(x+t)^2(x-t)$$
$$(4xt + 2t^2 + 2x^2)u_{\zeta\eta} = 4(x^2 + t^2 + 2xt)(x-t)$$
$$u_{\zeta\eta} = 2\zeta$$</td></tr>
<tr><td colspan="2">Integrate this normal form</td></tr>
<tr><td>Integrate wrt η
$$u_\zeta = fn(\zeta)$$
Integrate wrt ζ
$$u = f(\zeta) + g(\eta)$$
$$u(x,t) = f(x^2 + t^2) + g(x^2 - t^2)$$</td><td>Integrate wrt ζ
$$u_\eta = \frac{2\zeta^2}{2} + fn((\eta) = \zeta^2 + fn(\eta)$$
Integrate wrt η
$$u = \zeta^2\eta + f(\eta) + g(\zeta)$$
$$u = (x-t)^2(x^2+t^2) + f(x-t) + g(x^2+t^2)$$</td></tr>
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Second Year	Third Year
Introduce Cauchy Data	

Second Year

$$u(x,1) = f(x^2+1) + g(x^2-1) = x^4 \tag{3}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial \zeta}{\partial t} f'(x^2+t) + \frac{\partial \eta}{\partial t} g'(x^2-t)$$
$$= 2tf'(x^2+t) - 2tg'(x^2-t)$$

$$\frac{\partial u}{\partial t}(x,1) = 2f'(x^2+1) - 2g'(x^2-1) = 0$$

$$f'(x^2+1) - g'(x^2-1) = 0 \tag{4}$$

Now, take partial derivative of (3) w.r.t. x

$$\left.\frac{\partial u}{\partial x}\right|_{t=1} = \frac{\partial \zeta}{\partial x} f'(x+1) + \frac{\partial \eta}{\partial x} g'(x^2-1) = 4x^3$$

$$\Rightarrow 2xf'(x+1) + 2xg'(x^2-1) = 4x^3$$
$$\Rightarrow f'(x+1) + g'(x^2-1) = 2x^2 \tag{5}$$

Adding (4) and (5) produces

$$f' = x^2$$

so

$$\frac{\partial x}{\partial \zeta}\frac{\partial f}{\partial x} = x^2 \Rightarrow \frac{1}{2x}\frac{\partial f}{\partial x} = x^2 \Rightarrow \frac{\partial f}{\partial x} = 2x^3$$

$$\Rightarrow f(x,t) = \frac{x^4}{2} + \phi(t) \tag{6}$$

Likewise, subtracting (4) from (5) produces

$$g' = x^2$$

so

$$\frac{\partial x}{\partial \eta}\frac{\partial g}{\partial x} = x^2 \Rightarrow \frac{1}{2x}\frac{\partial g}{\partial x} = x^2 \Rightarrow \frac{\partial g}{\partial x} = 2x^3$$

$$\Rightarrow g(x,t) = \frac{x^4}{2} + \psi(t) \tag{7}$$

Adding (6) and (7) and comparing with (3) $\Rightarrow \phi(t) + \psi(t) = 0$. Therefore the solution to the Cauchy Problem is

$$u(x,t) = x^4$$

Third Year

$$u(x,0) = (x^2)(x^2) + f(x) + g(x^2) = -\frac{1}{3}x^3 \tag{8}$$

and

$$\frac{\partial u}{\partial t} = (x-t)^2(2t) + (x^2+t^2)2(x-t)(-1) + \frac{\partial \zeta}{\partial t} f'(x-t)$$
$$+ \frac{\partial \eta}{\partial t} g'(x^2+t^2)$$

$$\frac{\partial u}{\partial t}(x,0) = (x^2)(2)(x)(-1) - f'(x) = -2x^3 - f'(x) = x^2$$

$$\Rightarrow f'(x) = -2x^3 - x^2$$

$$f(x) = -\frac{2x^4}{4} - \frac{x^3}{3} + A = -\frac{x^4}{2} - \frac{x^3}{3} + A$$

and

$$g(x^2) = -\frac{1}{3}x^3 - x^4 - f(x)$$
$$= -\frac{1}{3}x^3 - x^4 + \frac{x^4}{2} + \frac{x^3}{3} - A = -\frac{x^4}{2} - A$$

ie

$$g(\lambda) = -\frac{\lambda^2}{2} - A$$

So solution to Cauchy Problem is

$$u(x,t) = (x-t)^2(x^2+t^2) + \left(-\frac{(x-t)^4}{2} - \frac{(x-t)^3}{3} + A\right)$$
$$+ \left(-\frac{(x^2+t^2)^2}{2} - A\right)$$

$$= (x-t)^2(x^2+t^2) - \frac{(x-t)^4}{2} - \frac{(x-t)^3}{3} - \frac{(x^2+t^2)^2}{2}$$