Exam Questions, Newcastle University

(which Newcastle University claim are "not even remotely similar")

Second Year	Third Year
Given Equation	
$$\frac{y^3}{3x^2}\frac{\partial z}{\partial x}+\frac{e^{2z}}{y^2}\frac{\partial z}{\partial y}=e^{-z}$$	$$5y^2\frac{\partial z}{\partial x}+x^2\frac{\partial z}{\partial y}=2x^5y$$ Re-arrange this equation $$\frac{5y^2}{x^2}\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=2x^3y$$
Find the characteristic congruence	
$$\dot{x}=\frac{y^3}{3x^2},\quad \dot{y}=\frac{e^{2z}}{y^2},\quad \dot{z}=e^{-z}$$ From the last eqn. $$\int e^z dz=\int dt$$ $$e^z=t+A$$ Set $z(1)=0\Rightarrow A=0$, so $$\underline{e^z=t}$$ So $e^{2z}=t^2$, and second eqn. becomes $$\dot{y}=\frac{t^2}{y^2}$$ $$\int y^2dy=\int t^2dt$$ $$\underline{\frac{y^3}{3}=\frac{t^3}{3}+\lambda}$$ Finally $$\dot{x}=\frac{t^3+3\lambda}{3x^2}$$ $$3\int x^2dx=\int(t^3+3\lambda)dt$$ $$\underline{x^3=\frac{t^4}{4}+3\lambda t+\mu}$$	$$\dot{x}=\frac{5y^2}{x^2},\quad \dot{y}=1,\quad \dot{z}=2x^3y$$ From the second equation $$y=t+A$$ Let $y(0)=0\Rightarrow A=0$, so $$\underline{y=t}$$ The first equation becomes $$\dot{x}=\frac{5t^2}{x^2}$$ $$\Rightarrow\int x^2dx=5\int t^2dt$$ $$\frac{x^3}{3}=\frac{5t^3}{3}+B\Rightarrow\underline{x^3=5t^3+\lambda}$$ Finally $$\dot{z}=2x^3y$$ $$=2(5t^3+\lambda)t$$ $$=10t^4+2\lambda t$$ $$\Rightarrow\underline{z=2t^5+\lambda t^2+\mu}$$

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Eliminate t and find λ, μ as functions of x, y, z	
$$\lambda = \frac{1}{3}(y^3 - t^3) = \underline{\frac{1}{3}(y^3 - e^{3z})}$$ $$\mu = x^3 - \frac{t^4}{4} - 3\lambda t$$ $$\mu = x^3 - \frac{e^{4z}}{4} - 3\left\{\frac{1}{3}\left(y^3 - e^{3z}\right)\right\}e^z$$ $$\mu = x^3 - \frac{e^{4z}}{4} - e^z y^3 + e^{4z}$$ $$\underline{\mu = x^3 - \frac{3e^{4z}}{4} - e^z y^3}$$	$$\lambda = x^3 - 5t^3$$ $$\Rightarrow \underline{\lambda = x^3 - 5y^3}$$ $$\mu = z - 2t^5 - \lambda t^2$$ $$\mu = z - 2y^5 - (x^3 - 5y^3)y^2$$ $$\mu = z - 2y^5 - x^3y^2 + 5y^5$$ $$\underline{\mu = z + 3y^5 - x^3y^2}$$
General Solution	
The General Solution is given by $$f(\lambda, \mu) = 0 \Rightarrow f\left(\frac{1}{3}(y^3 - e^{3z}), x^3 - \frac{3e^{4z}}{4} - e^z y^3\right) = 0$$	The General Solution is given by $$f(\lambda,\ \mu) = 0 \Rightarrow \underline{f\left(x^3 - 5y^3,\ \ z + 3y^5 - x^3y^2\right) = 0}$$
Insert Cauchy Data	
First convert General Solution to the form $$x^3 - \frac{3e^{4z}}{4} - e^z y^3 = g\left(\frac{1}{3}(y^3 - e^{3z})\right)$$ When $z = 0,\ \ x = \tau,\ \ y = \tau$ $$\tau^3 - \frac{3}{4} - \tau^3 = g\left(\frac{1}{3}(\tau^3 - 1)\right) \Rightarrow g(\zeta) = -\frac{3}{4}$$ Therefore $$x^3 - \frac{3e^{4z}}{4} - e^z y^3 = -\frac{3}{4}$$	
To find solutions thru $(1, 1, 0)$	
For a solution to contain $(1, 1, 0)$ $$1 - \frac{3}{4} - 1 = g\left(\frac{1}{3}(1 - 1)\right) \Rightarrow -\frac{3}{4} = g(0)$$ For two particular solutions take 1. $$g(\zeta) = -\frac{3}{4} \qquad (1)$$ 2. $$g(\zeta) = -\frac{3}{4} + \zeta \qquad (2)$$ (1) produces $$\underline{x^3 - \frac{3e^{4z}}{4} - e^z y^3 = -\frac{3}{4}}$$ (2) produces $$\underline{x^3 - \frac{3e^{4z}}{4} - e^z y^3 = -\frac{3}{4} + \frac{1}{3}(y^3 - e^{3z})}$$	